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Minimum *r*-neighborhood covering set of permutation graphs

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For a connected graph $G(V, E)$ and a fixed integer $r > 0$, a node $q \in V$ r-dominates another node $s \in V$ if $d(q, s) \leq r$. An edge (q, s) is r-neighborhood covered by a vertex t, if $d(q, t) \leq r$ and $d(s, t) \leq r$, i.e., both the vertices q and s are r-dominated by the vertex t. A set $C_r \subseteq V$ is known to be a r-neighborhood covering (r-NC) set of graph G if and only if one or more vertices of C_r r-dominate each edge in E. Among all r-NC sets of graph G, the set with fewest cardinality is the minimum r-NC set of G and we indicate its cardinality as r-NC-number and we denote it by the symbol $\rho(G, r)$. This is an NP-complete problem on general graphs. It is also NP-complete for chordal graphs. Here, we develop an $O(n)$ time algorithm for computing a minimum r-NC set of permutation graphs, where n indicates the order of the set V.

Keywords: r-neighborhood covering; permutation graph; algorithm.

Mathematics Subject Classification 2020: 05C30, 05C12, 68R10

1. Introduction

Permutation graph [5] is one of the vital intersection graphs. It has more complex properties than several special class of subgraphs like trees, interval graphs, circular-arc graphs, etc. It is a sub-class of comparability graphs [12] and trapezoid graphs. Each permutation graph has a unique matching diagram or permutation representation [5]. A matching diagram of a permutation graph having n vertices has two arrays — array of numbers, denoted by i and array of permutation numbers, denoted by $\pi(i)$. If the adjacency list or an adjacency matrix of a permutation graph is given then its matching diagram can be built (see [5, 10]) in $O(n^2)$ time. On the other hand, the two arrays of a matching diagram can be stored in computer memory in just $O(n)$ time. Because of that, we suppose that a permutation representation is provided as the input graph. A permutation graph and its corresponding matching diagram are drawn separately in Figs. 2 and 1, respectively.

An important variation of domination problem is the r-neighborhood covering (r-NC) problem. Domination is a popular and effective model for solving location-problems in the field like networking, operation research, etc. Our considered graph $G(V, E)$ is simple, finite, connected and undirected. We know that a *path* is an alternative finite sequence of nodes and edges of G. The number of edges belong to a path indicates the *length* of that path. Here, we use the symbol $d(v_1, v_2)$ to

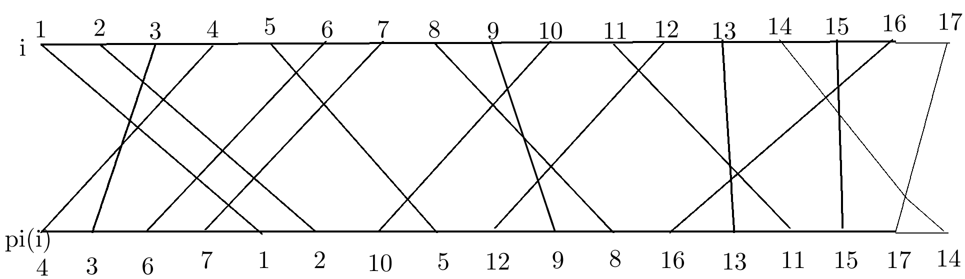

Fig. 1. Matching diagram of Fig. 2.

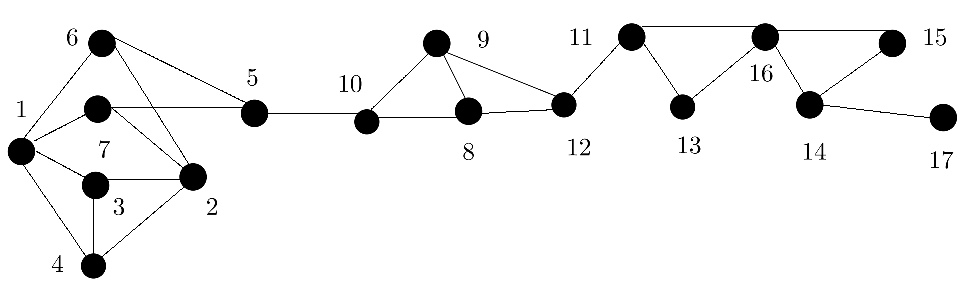

Fig. 2. A permutation graph G.

indicate the *distance* between two node points v_1 and v_2 and we take the minimum length of all paths between them as $d(v_1, v_2)$.

A node point $s \in V$ is r-dominated by a node $q \in V$ if $d(q, s) \leq r$. An edge (q, s) is r-neighborhood covered (r-NCOV) by a vertex t if $d(q, t) \leq r$ and $d(s, t) \leq r$ i.e., both node points q and s are r-dominated by the node point t. A set $C_r \subseteq V$ is known to be a r-NC set of graph G iff one or more vertices of C_r r-dominate each edge in E. Among all r-NC sets of graph G, the set with fewest cardinality is the minimum r-NC set of G and we indicate its cardinality as r-NC-number and we denote it by the symbol $\rho(G, r)$. This is an NP-complete problem on general graphs. In this paper, we develop an optimal algorithm for solving this problem on our considered graph.

1.1. *Literature review*

The r-NC-problem is a special kind of domination problems. Many researchers studied thoroughly this problem on different graph classes. For interval graphs, Lehel *et al.* [8] developed an algorithm in linear time to evaluate $\rho(G, 1)$. Also, linear-time algorithms are available [3, 7] for evaluating $\rho(G, 1)$ on strongly chordal graphs. Later for chordal graphs, Hwang and Chang [7] verified that this problem is NP-complete. Again, an $O(n)$ time algorithm is available (see [9]) for solving 2-NC problem on interval graphs. After that Ghosh *et al.* [4] formulated an algorithm to determine 2-NC set on trapezoid graphs that runs in $O(n)$ time. Besides these Barman *et al.* [2] solved the r-NC-problem on interval graphs in $O(n)$ time, for fixed $r \in N$. Recently, on permutation graph G, Rana *et al.* [13] developed an algorithm to solve 2-NC problem that takes $\mathrm{O}(n + \bar{m})$ time with n nodes and m edges, and $\bar{m}$ indicates the order of $E(\bar{G})$.

1.2. *Applications*

The r-NC-problem is very interesting section in graph theory. Some real applications of r-NC-problem are found in biological network modeling, facility location problems, survey of land, communication related networks, kernels of games [6], coding theory, etc. On the other hand, in scheduling problem, permutation graph has so many applications. At some definite time duration, for describing storage demands of a number of programs, we can also use permutation graphs [5]. A real life example of permutation graph is Flight-Altitudes problem. Let there are two collections of cities and airports. We can assign flight-altitudes for connecting cities and prevent the intersecting flights from colliding in the mid-air.

1.3. *Main outcome*

The main outcome of our paper is focused to develop an $O(n)$ time algorithm to obtain a minimum-r-NC set of a simple, connected and undirected permutation-graph.

1.4. *Arrangement of this paper*

We explain the formation of Breadth-First Search (BFS)-tree T_1^* of our considered graph in the following section. In Sec. 2, notations needed for this work are discussed too. Section 3 contains some vital results associated to the r-NC set of permutation graph. In Sec. 4, we formulate our proposed algorithm for evaluating a minimum r-NC set of permutation graph and discuss about its time complexity.

2. BFS-Tree

BFS is one of the popular graph tour techniques. This search technique always forms a BFS-tree. In BFS technique, we first choose an arbitrary vertex $v \in V$ and placed it at level 0 as the root of the tree. Then we add all the edges of the graph G which are incident to the vertex v such that the addition of edges does not make any cycle. The new vertices added at this stage become the vertices at level 1 in the tree. Next, for each vertex at level 1, we add each new edge of G incident to this vertex to the tree as long as it does not make any cycle. The children of each vertex at level 1 are placed at level 2. Continue the same procedure until all the vertices of G have been added to the tree. Finally, we will get a BFS-tree of G.

For general graphs, making of the BFS-tree can be finished within $O(n+m)$ time [14], where n and m are usual notations. Later, Olariu [11] built that tree (familiar as *interval-tree*) on interval graphs. Mondal *et al.* [9] formulated an algorithm for creating the same tree (denoted by $T^*(i)$) for trapezoid-graphs within $O(n)$ time. Furthermore, Barman *et al.* [1] gave an algorithm (named as *PBFS*) to create the same tree ($T^*(x)$, where node point x is placed as root) for permutation-graphs which compiled in $O(n)$ time. We denote this tree $T^*(x)$ by the symbol T_x^*. The BFS-tree T_1^* of the permutation graph (see Fig. 2) is drawn in Fig. 3.

2.1. *Principal-path on* T_1^*

Suppose H indicates the height of T_1^* and q is a node point located at the Hth level of T_1^*. Obviously, $d(1,q) = H$. Now, we use a new terminology '*principal-path*' to indicate the shortest route/path between q and 1. If we use the symbol $p\text{node}(u)$ to represent the parent node of u, then the principal path will be $q \to p\text{node}(q) \to p\text{node}(p\text{node}(q)) \to \cdots \to 1$. We also denote the node which lying at kth level on the principal-path by the symbol c_k^*.

2.2. *Notations*

We make here a list of notations that are needful throughout our work.

c_j^* : c_j^* is the node at jth level on the principal-path of T_1^*.
L_j : L_j is the set of elements at jth level on T_1^*.
A_j : $A_j = L_j - \{c_j^*\}$.
H : The height of T_1^*.
$d(x,t)$: The shortest distance between two vertices x and t.

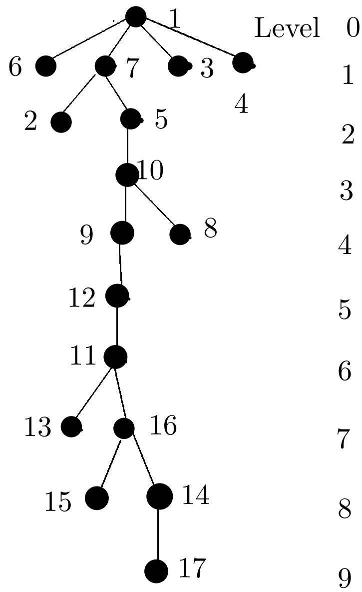

Fig. 3. BFS-tree T_1^* of Fig. 2.

3. Results Associated to r-NC Set of Permutation Graphs

We state and prove the following results.

Lemma 1. *For all* $g \in \bigcup_{k_1=0}^{r-1} L_{k_1}$, $d(c_{r-1}^*, g) \leq r$.

Proof. Obviously, $d(c_{k_1}^*, c_{r-1}^*) \leq (r-1-k_1) \leq r$ (as $c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{r-1}^*$), for $k_1 = 0, 1, 2, \ldots, (r-2)$. If g_{k_1} be any element of A_{k_1}, for $k_1 = 1, 2, \ldots, (r-1)$, then, $d(c_{r-1}^*, g_{k_1}) \leq (r+1-k_1) \leq r$ (as $c_{r-1}^* \to c_{r-2}^* \to \cdots \to c_{k_1-1}^* \to g_{k_1}$ or $c_{r-1}^* \to c_{r-2}^* \to \cdots \to c_{k_1}^* \to g_{k_1}$ or $c_{r-1}^* \to c_{r-2}^* \to \cdots \to c_{k_1+1}^* \to g_{k_1}$, etc.), for $k_1 = 1, 2, \ldots, (r-1)$. Hence, $d(c_{r-1}^*, g) \leq r$, $\forall\, g \in \bigcup_{k_1=0}^{r-1} L_{k_1}$. □

Observing this result, we have reached to the conclusion stated below.

Corollary 1. c_{r-1}^* *is a possible member of* r*-NC set.*

Proof. Watching out the 1st result, we are able to prove that for each edge (g, s) belongs to E, $g, s \in \bigcup_{k_1=0}^{r-1} L_{k_1}$ is r-NCOV by c_{r-1}^*. Hence the result follows. □

Lemma 2. *If* $A_1 = \emptyset$ *or each element of* A_1 *is adjacent with* c_1^* *or* c_2^*, *then for all* $g \in \bigcup_{k_1=0}^{r} L_{k_1}$, $d(c_r^*, g) \leq r$.

Proof. Suppose that g_{k_1} is an arbitrary element of A_{k_1}, for $k_1 = 1, 2, \ldots, H$. Now, it is clear to us that $(c_{k_1}^*, c_{k_1+1}^*) \in E$, for $k_1 = 0, 1, 2, \ldots, H$. Again, $d(c_r^*, c_0^*) = r$ (the length of the path $c_0^* \to c_1^* \to \cdots \to c_r^*$).

Case 1. When $A_1 = \emptyset$.

Here, $d(g_{k_1}, c_r^*) \le (r+2-k_1) \le r$ (observing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_r^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_r^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_r^*$), for $k_1 = 2, 3, \ldots, r$. Therefore, $d(c_r^*, g) \le r$, $\forall\, g \in \bigcup_{k_1=0}^{r} L_{k_1}$.

Case 2. When $(g_1, c_1^*) \in E$ or $(g_1, c_2^*) \in E$, $\forall\, g_1 \in A_1$.

In this case, $d(g_1, c_r^*) \le r$ (see the paths $g_1 \to c_1^* \to c_2^* \to \cdots \to c_r^*$ or $g_1 \to c_2^* \to c_3^* \to \cdots \to c_r^*$). Also, $d(g_{k_1}, c_r^*) \le (r+2-k_1) \le r$ (see the paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_r^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_r^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_r^*$), for $k_1 = 2, 3, \ldots, r$. So, $d(c_r^*, g) \le r$, $\forall\, g \in \bigcup_{k_1=0}^{r} L_{k_1}$. Hence the result follows. □

Observing this result, we have reached to the conclusion stated below.

Corollary 2. *If $A_1 = \emptyset$ or each element of A_1 is adjacent with c_1^* or c_2^*, then c_r^* be a possible member of r-NC set.*

Proof. Watching out the last result, we are able to prove that for each edge (g, h) belongs to E, for all $g, h \in \bigcup_{k_1=0}^{r} L_{k_1}$ is r-NCOV by c_r^*. Hence the result follows. □

Lemma 3. *If at any situation, c_p^* is chosen as an element of r-NC set, then for all $g \in \bigcup_{k_1=1}^{r} L_{p+k_1}$, $d(g, c_p^*) \le r$.*

Proof. Suppose that at any situation, we choose c_p^* as a current element of C_r and g_{p+k_1} is an arbitrary element of A_{p+k_1}, where $k_1 = 1, 2, \ldots, r$. Obviously $(c_{p+k_1-1}^*, g_{p+k_1}) \in E$. Now, $d(g_{p+k_1}, c_p^*) = k_1 \le r$ (observing this path $c_p^* \to c_{p+1}^* \to c_{p+2}^* \to \cdots \to c_{p+k_1-1}^* \to g_{p+k_1}$), for $k_1 = 1, 2, 3, \ldots, r$. Therefore, $d(g, c_p^*) \le r$, for all $g \in \bigcup_{k_1=1}^{r} L_{p+k_1}$. □

Lemma 4. *If at any situation, c_p^* is chosen as an element of r-NC set, then each edge $(g, h) \in E$, $g, h \in \cup_{k_1=p-(r-2)}^{p+r} L_{k_1}$ is r-NCOV by c_p^*.*

Proof. Suppose that $g_{k_1} \in A_{k_1}$, for $k_1 = 1, 2, \ldots, H$ and c_p^* is the current chosen element of r-NC set at any situation. Obviously, $d(c_p^*, c_{p-(r-2)}^*) = r-2 < r$ (see the path $c_p^* \to c_{p-1}^* \to \cdots \to c_{p-(r-2)}^*$) and $d(c_p^*, c_{p+r}^*) = r$ (observing the path $c_p^* \to c_{p+1}^* \to \cdots \to c_{p+r}^*$). Now, for $k_1 = p-(r-2), p-(r-3), \ldots, p$, $d(c_p^*, g_{k_1}) = p - (k_1 - 2) \le r$ (observing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_p^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_p^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_p^*$). Also, for $k_1 = p+1, p+2, \ldots, p+r$, $d(c_p^*, g_{k_1}) = k_1 - p \le r$ (see the path $g_{k_1} \to c_{k_1-1}^* \to c_{k_1-2}^* \to \cdots \to c_p^*$). So, $d(c_p^*, g_{k_1}) \le r, \forall\, g_{k_1} \in \cup_{k_1=p-(r-2)}^{p+r} L_{k_1}$. Therefore, each edge $(g, h) \in E, g, h \in \cup_{k_1=p-(r-2)}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. □

Lemma 5. *The node g_{k_1} may or may not adjacent to $c_{k_1}^*$, $\forall\, g_{k_1} \in A_{k_1}$, $k_1 = 1, 2, \ldots, H$.*

Proof. It is obvious. □

Lemma 6. *If at any situation, c_p^* is chosen as current element of r-NC set and c_{p+2r-2}^* exists, then c_{p+2r-2}^* be a possible member of r-NC set.*

Proof. We suppose that c_p^* is chosen as a current element of r-NC set at any situation and c_{p+2r-2}^* exists. Also, we suppose that $g_{k_1} \in A_{k_1}$, where $k_1 = 1, 2, \ldots, H$. So, using the result of Lemma 4, we can write that each edge $(g, h) \in E$, where $g, h \in \bigcup_{k_1=1}^{r} L_{p+k_1}$ is r-NCOV by c_p^*. Now, we have to prove that every edge $(g, h) \in E$, where $g, h \in \bigcup_{k_1=p+r}^{p+2r-2} L_{k_1}$ is r-NCOV by c_{p+2r-2}^*. It is clear that $d(c_{p+r}^*, c_{p+2r-2}^*) = r - 2 \le r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r-2}^*$). Now, for $k_1 = p+r, p+r+1, \ldots, p+2r-2, d(c_{p+2r-2}^*, g_{k_1}) = p+2r-k_1 \le r$ (observing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_{p+2r-2}^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{p+2r-2}^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_{p+2r-2}^*$). So, $d(c_{p+2r-2}^*, g_{k_1}) \le r, \forall g_{k_1} \in \bigcup_{k_1=p+r}^{p+2r-2} L_{k_1}$. Therefore, each edge $(g, h) \in E, g, h \in \bigcup_{k_1=p+r}^{p+2r-2} L_{k_1}$ is r-NCOV by c_{p+2r-2}^*, i.e., c_{p+2r-2}^* is a possible member of r-NC set. □

Lemma 7. *If at any situation, c_p^* is chosen as a current element of r-NC set and $A_{p+r} = \emptyset$ and c_{p+2r-1}^* exists, then c_{p+2r-1}^* be a possible member of r-NC set.*

Proof. Suppose that c_p^* is chosen as a current element of r-NC set at any situation and $g_{k_1} \in A_{k_1}$, for $k_1 = 1, 2, \ldots, H$. Also, we presume that c_{p+2r-1}^* exists. So, using the result of Lemma 4, we can decide that every edge $(g, h) \in E$, where $g, h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. Also, $d(c_{p+r}^*, c_{p+2r-1}^*) = (r-1) < r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r-1}^*$). Again, let A_{p+r} be an empty set. In this case, for $k_1 = (p+r+1), (p+r+2), \ldots, (p+2r-1)$, $d(c_{p+2r-1}^*, g_{k_1}) = (p+2r-k_1+1) \le r$ (observing the paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_{p+2r-1}^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{p+2r-1}^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_{p+2r-1}^*$). So, we can write that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=p+r}^{p+2r-1} L_{k_1}$ is r-NCOV by c_{p+2r-1}^*. Hence the result follows. □

Lemma 8. *If c_p^* is chosen as a current element of r-NC set at any situation and if $(g, c_{p+r}^*) \in E$, $\forall\, g \in A_{p+r}$ and c_{p+2r-1}^* exists, then c_{p+2r-1}^* be a possible member of r-NC set.*

Proof. We suppose that c_p^* is chosen as a current element of r-NC set at any situation and $g_{k_1} \in A_{k_1}$, where $k_1 = 1, 2, \ldots, H$. Also, we presume that c_{p+2r-1}^* exists. So, applying the result of Lemma 4, we can decide that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. Also, $d(c_{p+r}^*, c_{p+2r-1}^*) = (r-1) < r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r-1}^*$). Let $(g, c_{p+r}^*) \in E$, $\forall\, g \in A_{p+r}$. In this case, $\forall\, g_{p+r} \in A_{p+r}$, $d(c_{p+2r-1}^*, g_{p+r}) \le r$ (observing the paths $g_{p+r} \to c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r-1}^*$ or $g_{p+r} \to c_{p+r+1}^* \to c_{p+r+2}^* \to \cdots \to c_{p+2r-1}^*$). Also, for $k_1 = (p+r+1), (p+r+2), \ldots, (p+2r-1)$, $d(c_{p+2r-1}^*, g_{k_1}) = (p+2r-k_1+1) \le r$

(observing the paths $g_{k_1} \to c^*_{k_1-1} \to c^*_{k_1} \to \cdots \to c^*_{p+2r-1}$ or $g_{k_1} \to c^*_{k_1} \to c^*_{k_1+1} \to \cdots \to c^*_{p+2r-1}$ or $g_{k_1} \to c^*_{k_1+1} \to c^*_{k_1+2} \to \cdots \to c^*_{p+2r-1}$). So, we can write that every edge $(g,h) \in E$, where $g,h \in \bigcup_{k_1=p+r}^{p+2r-1} L_{k_1}$ is r-NCOV by c^*_{p+2r-1}. Hence, c^*_{p+2r-1} is a possible member of r-NC set. □

Lemma 9. *If c^*_p is chosen as a current element of r-NC set at any situation and if $(g, c^*_{p+r+1}) \in E$ but $(g, c^*_{p+r}) \notin E$, $\forall\, g \in A_{p+r}$ and c^*_{p+2r-1} exists, then c^*_{p+2r-1} be a possible member of r-NC set.*

Proof. Suppose that c^*_p is chosen as a current element of r-NC set C_r at any situation and $g_{k_1} \in A_{k_1}$, where $k_1 = 1, 2, \ldots, H$. Also, we presume that c^*_{p+2r-1} exists. So, applying Lemma 4, we can write that every edge $(g,h) \in E$, where $g,h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c^*_p. Also, $d(c^*_{p+r}, c^*_{p+2r-1}) = (r-1) < r$ (watching out the path $c^*_{p+r} \to c^*_{p+r+1} \to \cdots \to c^*_{p+2r-1}$). Also, let $(g, c^*_{p+r+1}) \in E$ but $(g, c^*_{p+r}) \notin E$, $\forall\, g \in A_{p+r}$. Here, $\forall\, g_{p+r} \in A_{p+r}$, $d(c^*_{p+2r-1}, g_{p+r}) = (r-1)$ (see the path $g_{p+r} \to c^*_{p+r+1} \to c^*_{p+r+2} \to \cdots \to c^*_{p+2r-1}$). Also, for $k_1 = (p+r+1), (p+r+2), \ldots, (p+2r-1)$, $d(c^*_{p+2r-1}, g_{k_1}) = (p+2r-k_1+1) \le r$ (observing these paths $g_{k_1} \to c^*_{k_1-1} \to c^*_{k_1} \to \cdots \to c^*_{p+2r-1}$ or $g_{k_1} \to c^*_{k_1} \to c^*_{k_1+1} \to \cdots \to c^*_{p+2r-1}$ or $g_{k_1} \to c^*_{k_1+1} \to c^*_{k_1+2} \to \cdots \to c^*_{p+2r-1}$). So, we can decide that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p+r}^{p+2r-1} L_{k_1}$ is r-NCOV by c^*_{p+2r-1}. Hence the result follows. □

Lemma 10. *If c^*_p is chosen as a current element of r-NC set at any situation and $A_{p+r} = \emptyset$ and $A_{p+r+1} = \emptyset$, then c^*_{p+2r} be a possible member of r-NC set.*

Proof. We assume that c^*_p is chosen as a current element of r-NC set at any situation and g_{k_1} is an arbitrary element of A_{k_1}, where $k_1 = 1, 2, \ldots, H$. Also, we presume that c^*_{p+2r} exists, and A_{p+r} and A_{p+r+1} are both empty sets. Now, applying Lemma 4, we can write that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c^*_p. Furthermore, $d(c^*_{p+r}, c^*_{p+2r}) = r$ (see the path $c^*_{p+r} \to c^*_{p+r+1} \to \cdots \to c^*_{p+2r}$). Since, $A_{p+r} = \emptyset$ and $A_{p+r+1} = \emptyset$ then there is only one edge $(c^*_{p+r}, c^*_{p+r+1}) \in E$ among the elements of L_{p+r} and L_{p+r+1}. Also, $\forall\, k_1 = (p+r+2), (p+r+3), \ldots, (p+2r)$, $d(c^*_{p+2r}, g_{k_1}) = (p+2r-k_1+2) \le r$ (see these paths $g_{k_1} \to c^*_{k_1-1} \to c^*_{k_1} \to \cdots \to c^*_{p+2r}$ or $g_{k_1} \to c^*_{k_1} \to c^*_{k_1+1} \to \cdots \to c^*_{p+2r}$ or $g_{k_1} \to c^*_{k_1+1} \to c^*_{k_1+2} \to \cdots \to c^*_{p+2r}$). So, we can decide that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p+r}^{p+2r} L_{k_1}$ is r-NCOV by c^*_{p+2r}. Hence the result follows. □

Lemma 11. *If c^*_p is chosen as a current element of r-NC set at any situation and if $(g, c^*_{p+r+1}) \in E$, $\forall\, g \in A_{p+r} \cup A_{p+r+1}$ and c^*_{p+2r} exists, then c^*_{p+2r} be a possible member of r-NC set.*

Proof. We suppose that c^*_p is chosen as a current element of r-NC set at any situation and g_{k_1} is an arbitrary element of A_{k_1}, where $k_1 = 1, 2, \ldots, H$. Also, we presume that $(g, c^*_{p+r+1}) \in E$, $\forall\, g \in A_{p+r} \cup A_{p+r+1}$ and c^*_{p+2r} exists. Now, applying

Lemma 4, we can write that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. Also, $d(c_{p+2r}^*, c_{p+r}^*) = r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r}^*$). Again, $d(c_{p+2r}^*, g_{p+r}) = r$ (see the path $g_{p+r} \to c_{p+r+1}^* \to c_{p+r+2}^* \to \cdots \to c_{p+2r}^*$), $\forall\, g_{p+r} \in A_{p+r}$. Furthermore, $\forall\, g_{p+r+1} \in A_{p+r+1}$, $d(c_{p+2r}^*, g_{p+r+1}) \le r$ (observing these paths $g_{p+r+1} \to c_{p+r+1}^* \to c_{p+r+2}^* \to \cdots \to c_{p+2r}^*$ or $g_{p+r+1} \to c_{p+r+2}^* \to c_{p+r+3}^* \to \cdots \to c_{p+2r}^*$). Besides these, for $k_1 = (p+r+2), (p+r+3), \ldots, (p+2r)$, $d(c_{p+2r}^*, g_{k_1}) = (p+2r-k_1+2) \le r$ (seeing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_{p+2r}^*$). So, we can write that each edge $(g,h) \in E, g,h \in \bigcup_{k_1=p+r}^{p+2r} L_{k_1}$ is r-NCOV by c_{p+2r}^*. Hence c_{p+2r}^* is a possible member of r-NC set. □

Lemma 12. *If c_p^* is chosen as a current element of r-NC set at any situation and $A_{p+r} = \emptyset$ and $(g, c_{p+r+1}^*) \in E$, $\forall\, g \in A_{p+r+1}$ and c_{p+2r}^* exists, then c_{p+2r}^* be a possible member of r-NC set.*

Proof. Suppose that c_p^* is chosen as a current element of r-NC set at any situation and g_{k_1} is an arbitrary element of A_{k_1}, where $k_1 = 1, 2, \ldots, H$. Also, we presume that A_{p+r} is an empty set and $(g, c_{p+r+1}^*) \in E$, $\forall\, g \in A_{p+r+1}$ and c_{p+2r}^* exists. Now, applying the result of Lemma 4, we can write that every edge $(g,h) \in E$, where $g,h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. Again, $d(c_{p+2r}^*, c_{p+r}^*) = r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r}^*$). Furthermore, $\forall\, g_{p+r+1} \in A_{p+r+1}$, $d(c_{p+2r}^*, g_{p+r+1}) \le r$ (see the paths $g_{p+r+1} \to c_{p+r+1}^* \to c_{p+r+2}^* \to \cdots \to c_{p+2r}^*$ or $g_{p+r+1} \to c_{p+r+2}^* \to c_{p+r+3}^* \to \cdots \to c_{p+2r}^*$). Besides these, for $k_1 = (p+r+2), (p+r+3), \ldots, (p+2r)$, $d(c_{p+2r}^*, g_{k_1}) = (p+2r-k_1+2) \le r$ (observing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_{p+2r}^*$). So, we can decide that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p+r}^{p+2r} L_{k_1}$ is r-NCOV by c_{p+2r}^*. Hence the result follows. □

Lemma 13. *If c_p^* is chosen as a current element of r-NC set at any situation and $A_{p+r+1} = \emptyset$ and $(g, c_{p+r+1}^*) \in E$, $\forall\, g \in A_{p+r}$ and c_{p+2r}^* exists, then c_{p+2r}^* be a possible member of r-NC set.*

Proof. Suppose that c_p^* is chosen as a current element of r-NC set at any situation and g_{k_1} is an arbitrary element of A_{k_1}, where $k_1 = 1, 2, \ldots, H$. Also, we presume that $A_{p+r+1} = \emptyset$ and $(g, c_{p+r+1}^*) \in E$, $\forall\, g \in A_{p+r}$ and c_{p+2r}^* exists. Now, with the help of Lemma 4, we can write that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c_p^*. Again, $d(c_{p+2r}^*, c_{p+r}^*) = r$ (see the path $c_{p+r}^* \to c_{p+r+1}^* \to \cdots \to c_{p+2r}^*$). Furthermore, $\forall\, g_{p+r} \in A_{p+r}$, $d(c_{p+2r}^*, g_{p+r}) = r$ (observing the path $g_{p+r} \to c_{p+r+1}^* \to c_{p+r+2}^* \to \cdots \to c_{p+2r}^*$). Besides these, $\forall\, k_1 = (p+r+2), (p+r+3), \ldots, (p+2r)$, $d(c_{p+2r}^*, g_{k_1}) = (p+2r-k_1+2) \le r$ (seeing these paths $g_{k_1} \to c_{k_1-1}^* \to c_{k_1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1}^* \to c_{k_1+1}^* \to \cdots \to c_{p+2r}^*$ or $g_{k_1} \to c_{k_1+1}^* \to c_{k_1+2}^* \to \cdots \to c_{p+2r}^*$). So, we can decide that every edge $(g,h) \in E, g,h \in \bigcup_{k_1=p+r}^{p+2r} L_{k_1}$ is r-NCOV by c_{p+2r}^*. Therefore c_{p+2r}^* is a possible member of r-NC set. □

Lemma 14. *If H is less than r then $c^*_{k_1} \in C_r$, where k_1 represents only one arbitrary element in the set $\{0, 1, 2, \ldots, H\}$.*

Proof. We suppose that $H < r$ and $g_{k_1} \in L_{k_1}$, $k_1 \in \{1, 2, \ldots, H\}$. Now, $\forall\, k_1 = 1, 2, \ldots, H, d(c^*_0, g_{k_1}) = k_1 < r$(see the path $c^*_0 \to c^*_1 \to \cdots \to c^*_{k_1-1} \to g_{k_1}$). So, each edge $(g, h) \in E, g, h \in \bigcup_{k_1=0}^{H} L_{k_1}$ is r-NCOV by c^*_0. Again, $\forall\, d(c^*_0, c^*_H) = H < r$(see the path $c^*_0 \to c^*_1 \to \cdots \to c^*_H$). Let h_{k_1} be an arbitrary element of $L_{k_1} - \{c^*_{k_1}\}$, where $k_1 = 1, 2, \ldots, H$. Also, $\forall\, k_1 = 1, 2, 3, \ldots, H$, $d(c^*_H, h_{k_1}) \leq (H + 2 - k_1) \leq r$ (seeing these paths $h_{k_1} \to c^*_{k_1-1} \to c^*_{k_1} \to \cdots \to c^*_H$ or $h_{k_1} \to c^*_{k_1} \to c^*_{k_1+1} \to \cdots \to c^*_H$ or $h_{k_1} \to c^*_{k_1+1} \to c^*_{k_1+2} \to \cdots \to c^*_H$). So, we can write that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=0}^{H} L_{k_1}$ is r-NCOV by c^*_H. Similarly, we can say that, every edge $(g, h) \in E, g, h \in \bigcup_{k_1=0}^{H} L_{k_1}$ is r-NCOV by other elements (excluding c^*_0, c^*_H) of the principal-path. Hence the result follows. □

Lemma 15. *If $H \in [r, 2r - 1]$, then $C_r = \{c^*_{r-1}\}$.*

Proof. Suppose that $H \in [r, 2r - 1]$ and $g_{k_1} \in A_{k_1}$, where $g_{k_1} \in L_{k_1}$, $k_1 \in \{1, 2, \ldots, H\}$. Now, applying the result of Corollary 1, we can write that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=0}^{r-1} L_{k_1}$ is r-NCOV by c^*_{r-1}. Again, using the result of Lemma 4, we can decide that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=r-1}^{2r-1} L_{k_1}$ is r-NCOV by c^*_{r-1}. So, we can write that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=0}^{2r-1} L_{k_1}$ is r-NCOV by c^*_{r-1}. Hence the result follows. □

Lemma 16. *If c^*_p is chosen as a current element of r-NC set at any situation and $H \in [p + r + 1, p + 2r - 3]$, then $c^*_{p+r+k_1}$ be a possible member of r-NC set, where k_1 represents only one arbitrary element in the set $\{1, 2, \ldots, r - 3\}$.*

Proof. We suppose that $p + r + 1 \leq H \leq p + 2r - 3$ and c^*_p is chosen as a current element of r-NC set at any situation. Also, we presume that $g_{k_1} \in L_{k_1}$, where $k_1 \in \{1, 2, 3, \ldots, H\}$. Now, with the help of Lemma 4, we can write that every edge $(g, h) \in E, g, h \in \bigcup_{k_1=p}^{p+r} L_{k_1}$ is r-NCOV by c^*_p. Now, $\forall\, k_1 = 2, 3, \ldots, r - 3$, $d(c^*_{p+r+1}, g_{p+r+k_1}) = (k_1 - 1) < r$ (see the path $c^*_{p+r+1} \to c^*_{p+r+2} \to \cdots \to c^*_{p+r+k_1-1} \to g_{p+r+k_1}$). Furthermore, $d(c^*_{p+r+1}, g_{p+r+1}) \leq 2$ (seeing the paths $g_{p+r+1} \to c^*_{p+r} \to c^*_{p+r+1}$ or $g_{p+r+1} \to c^*_{p+r+1}$). So, we can write that every edge $(g, z) \in E, g, z \in \bigcup_{k_1=p+r+1}^{p+2r-3} L_{k_1}$ is r-NCOV by c^*_{p+r+1}. Again, $d(c^*_{p+r+1}, c^*_{p+2r-3}) = (r - 4) < r$ (see the path $c^*_{p+r+1} \to c^*_{p+r+2} \to \cdots \to c^*_{p+2r-3}$).

Let z_{p+r+k_1} be an arbitrary element of $L_{p+r+k_1} - \{c^*_{p+r+k_1}\}$, where $k_1 = 0, 1, 2, 3, \ldots, r - 3$. Also, $\forall\, k_1 = 0, 1, 2, 3, \ldots, r - 4$, $d(c^*_{p+2r-3}, z_{p+r+k_1}) \leq (r - 1 - k_1) \leq r$ (observing these paths $z_{p+r+k_1} \to c^*_{p+r+k_1-1} \to c^*_{p+r+k_1} \to \cdots \to c^*_{p+2r-3}$ or $z_{p+r+k_1} \to c^*_{p+r+k_1} \to c^*_{p+r+k_1+1} \to \cdots \to c^*_{p+2r-3}$ or $z_{p+r+k_1} \to c^*_{p+r+k_1+1} \to c^*_{p+r+k_1+2} \to \cdots \to c^*_{p+2r-3}$). Furthermore, $d(c^*_{p+2r-3}, z_{p+2r-3}) \leq 2$ (seeing these paths $z_{p+2r-3} \to c^*_{p+2r-4} \to c^*_{p+2r-3}$ or $z_{p+2r-3} \to c^*_{p+2r-3}$). So, we can decide that every edge $(g, z) \in E, g, z \in \cup_{k_1=p+r}^{p+2r-3} L_{k_1}$ is r-NCOV by c^*_{p+2r-3}.

Similarly, we can say that, each edge $(g, z) \in E, g, z \in \cup_{k_1=p+r}^{p+2r-3} L_{k_1}$ is r-NCOV by other elements (excluding $c^*_{p+r+1}, c^*_{p+2r-3}$) of the principal-path. Hence the result follows. □

4. Algorithm and its Complexity

Method F-NXT-P(p, P)
//This procedure is for computation of the level P when c^*_P is the upcoming element of set C_r, whenever it is assumed that c^*_p is current chosen element of set C_r. Also, the array $c^*_{k_1}, k_1 = 0, 1, 2, 3, \ldots, H$, and the sets P_{k_1}, A_{k_1} are known globally. Here, $H > p + r$.//

Initially $P = p$;

if $A_{p+r} \cup A_{p+r+1} = \emptyset$ or if $(g, c^*_{p+r+1}) \in E$, $\forall g \in A_{p+r} \cup A_{p+r+1}$ or if $A_{p+r} = \emptyset$ and

$(g, c^*_{p+r+1}) \in E$, $\forall g \in A_{p+r+1}$ or if $A_{p+r+1} = \emptyset$ and $(g, c^*_{p+r+1}) \in E$, $\forall g \in A_{p+r}$ and c^*_{p+2r} exists, then $P = p + 2r$; (Lemmas 10–13)

else if $A_{p+r} = \emptyset$ or if $(g, c^*_{p+r}) \in E$, $\forall g \in A_{p+r}$ or if $(g, c^*_{p+r+1}) \in E$ but $(g, c^*_{p+r}) \notin E$,

$\forall g \in A_{p+r}$ and c^*_{p+2r-1} exists, then $P = p + 2r - 1$; (Lemmas 7–9)

else if c^*_{p+2r-2} exists, then $P = p + 2r - 2$; (Lemma 6)

else $P = p + r + k_1$, for any $k_1 = 1, 2, \ldots, r - 3$; (Lemma 16)

endif;

Turn back P;
end F-NXT-P

Observing the results discussed in Sec. 3, we can decide that c^*_{r-1} is always a possible element of r-NC set. Also, we have observed that if c^*_p is chosen as a current element of r-NC set at any situation, then there are three choices for selecting possible elements of r-NC set. Here, we can choose either c^*_{p+2r-2} or c^*_{p+2r-1} or c^*_{p+2r} as a new element of r-NC set depend upon some conditions. All the possible occurrences to choose the elements of r-NC set are meanwhile discussed in Sec. 3. Also, we develop a technique F-NXT-P in the above to compute the level P for upcoming element c^*_P of r-NC set C_r, if p, the level of the current chosen element c^*_p is available.

Here, we are representing the final algorithm for computing a minimum r-NC set C_r for permutation graphs with the help of the procedure F-NXT-P recurrently. The complete algorithm is represented as follows:

Algorithm MRNCSP
Given: Corresponding permutation representation k_1 and $\pi(k_1)$ of a permutation graph G, where $k_1 = 1, 2, 3, \ldots, n$.
Result: The r-NC set C_r (with fewest cardinality) of G.
At first set $C_r = \emptyset$ and $c_0^* = 1$, the root of T_1^*.
Step 1: Make BFS-tree T_1^* and compute its height H.
Step 2: Obtain the node points located on the principal-path of T_1^* and suppose these are $c_{k_1}^*, k_1 = 0, 1, 2, \ldots, H$.
Step 3: Determine the sets L_{k_1} and $A_{k_1}, k_1 = 0, 1, 2, \ldots, H$.
Step 4: If H is less than r then $C_r = C_r \cup \{c_p^*\}$, where p represents only one arbitrary element of $\{0, 1, 2, \ldots, H\}$; (Using Lemma 14)
else if $H \in [r, 2r-1]$ then $C_r = C_r \cup \{c_{r-1}^*\}$; (Using Lemma 15)
else if $A_1 = \emptyset$ or each element of A_1 is adjacent with c_1^* or c_2^* then
$p = r$; (Using Corollary 2)
else $p = r - 1$; (Using Corollary 1)
endif;
$C_r = C_r \cup \{c_p^*\}$.
Step 5: Repeat
Call F-NXT-P(p, P); //search level P for upcoming element of C_r//
$p = P$;
$C_r = C_r \cup \{c_p^*\}$;
Until ($|H - p| \leq r$);
end MRNCSP.

If we apply the Algorithm **MRNCSP** on the permutation graph G displayed in Fig. 2 for $r = 2$, then we get a minimum r-NC set $C_r = \{7, 12, 14\}$.

Lemma 17. *The r-NC set C_r is a minimum r-NC set.*

Proof. In Step 4 of our final Algorithm **MRNCSP**, we set up $C_r = \{c_p^*\}$, where p represents only one arbitrary element of the set $\{0, 1, 2, 3, \ldots, H\}$ depend upon the condition $H \leq r$ shown in Lemma 14. So, that r-NC set C_r is minimum as its cardinality is one. Again, if $H \in [r, 2r-1]$, then we set up $C_r = \{c_{r-1}^*\}$ applying the result of Lemma 15. So, C_r has again minimum number of elements for $H \in [r, 2r-1]$. Besides these, if $H \geq 2r$, then one of c_r^* and c_{r-1}^* will be a possible element of r-NC set. The node c_r^* to be a possible member of r-NC set depends upon the conditions of Corollary 2 and other node c_{r-1}^* to be a possible member of r-NC set depends upon the conditions of Corollary 1. In that case, we give our preference on c_r^* (if possible) to choose as the first element of r-NC set C_r, otherwise we choose c_{r-1}^* as the first element of C_r. Now, suppose c_p^* is chosen as a current element of r-NC set at any situation, then there are only three options for selecting possible elements of r-NC set. Either c_{p+2r-2}^* or c_{p+2r-1}^* or c_{p+2r}^* will be chosen as the new element of r-NC set at upcoming step depends upon some conditions. In these cases, we give our first preference to c^*_{p+2r}, second preference to c^*_{p+2r-1} and 3rd preference to c^*_{p+2r-2} in such a way that selected member r-neighborhood covers maximum number of edges. So, the final r-NC set C_r is a minimum r-NC set. □

Theorem 1. *The run time of the procedure* ***F-NXT-P*** *is just* $O(|\bigcup_{k_1=0}^{2r-3} L_{p+k_1}|) \approx O(n)$, *where* $|V| = n$.

Proof. It is clear to us that the sets $L_{k_1}, k_1 = 1, 2, 3, \ldots, H$ and the sets $A_{k_1}, k_1 = 1, 2, 3, \ldots, H$ are mutually-exclusive, that is $L_{k_1} \cap L_j = \emptyset$ and $A_{k_1} \cap A_j = \emptyset$, for $k_1 \neq j$ and $k_1, j = 1, 2, 3, \ldots, H$ and $L_{k_1} = A_{k_1} \cup \{c^*_{k_1}\}$, where $k_1 = 1, 2, 3, \ldots, H$. Also, we have observed that the nodes of the sets L_{p+k_1}, for $k_1 = 1, 2, 3, \ldots, 2r-3$ are taken into consideration for processing the sets used in the procedure **F-NXT-P**. Hence, the sum of the numbers of the nodes in these sets is $|\bigcup_{k_1=0}^{2r-3} L_{p+k_1}|$ and the induced subgraph $G(\bigcup_{k_1=0}^{2r-3} L_{p+k_1})$ is a part of T^*_1. Therefore, in that part of T^*_1, at most $|\bigcup_{k_1=0}^{2r-3} L_{p+k_1}| - 1$ edges may exist. Hence, the run time for computing the procedure **F-NXT-P** is $O(|\bigcup_{k_1=0}^{2r-3} L_{p+k_1}|) \approx O(n)$. □

Theorem 2. $O(n)$ *time is needed to run the algorithm* ***MRNCSP*** *for computation of a minimum r-NC set of a connected permutation graph* G, *where* $n = |V(G)|$.

Proof. For a given permutation representation, we can make T^*_1 in $O(n)$ time (see Step 1). Also, the height of T^*_1 can be computed in $O(n)$ time. In Step 2, we are also able to determine all the elements of the principal-path in just $O(n)$ time as there are only $H + 1 \leq n$ elements on principal-path. We are also able to evaluate set L_{k_1} and A_{k_1} in $O(n)$ time (see Step 3) as they are mutually exclusive. Again, for a given level p, only $O(|\bigcup_{k_1=0}^{2r-3} L_{p+k_1}|)$ time is needed to find the upcoming element by the procedure **F-NXT-P**. The algorithm **MRNCSP** calls the procedure **F-NXT-P** for $|C_r| - 1$ times. In each iteration, the value of p is grown by $2r$ or $2r-1$ or $2r-2$. So, Step 5 needs $O(|\bigcup_{k_1=0}^{H} L_{k_1}|) = O(n)$ time. Hence, $O(n)$ time is needed to run the algorithm **MRNCSP**. □

Theorem 3. *The algorithm* ***MRNCSP*** *can be compiled in computer consuming* $O(n)$ *space only, where* $n = |V|$.

Proof. There are two sets in a given permutation-representation. First one is the set of numbers $\{1, 2, 3, \ldots, n\}$ and second one is the collection of permutation-number $\pi(k_1), k_1 = 1, 2, 3, \ldots, n$. We can store these two sets in computer memory in $O(n)$ space. We also able to store the tree T^*_1 and the related sets $L_{k_1}, k_1 = 1, 2, 3, \ldots, H$ and the elements of principal-path $c^*_{k_1}, k_1 = 1, 2, 3, \ldots, H$ in $O(n)$ space. Again, $|C_r| \leq n$. Therefore, overall $O(n)$ space is required to run the Algorithm **MRNCSP**. □

5. Conclusion

The r-NC problem has been discussed briefly by many researchers and many real-world applications are found regarding this problem. Here, we solve this r-NC problem for a simple, connected and undirected permutation-graphs in $O(n)$ time. Our future planning is to apply our technique to compute r-NC set on Trapezoid-graphs, chordal graphs, etc.

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