2022

Mathematics

[Honours]

(B.Sc. Third Semester End Examination-2022)

PAPER-MTMH C302

(Group Theory – II and Linear Algebra-II)

Full Marks: 60 ***Time: 03 Hrs***

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[Use separate answer script for each group]

Group-A

[Group Theory-II]

1. Answer any TWO questions $2 \times 2 = 4$

(a) Show that an Abelian group of order 35 is cyclic.

(b) Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a(\neq 0) \in \mathbb{R} \right\}$

and $G^{1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$

Is $G \times G'$ forms a group ? Justify.

(c) Let G be a commutative group of order n. If $\gcd(m,n) = 1$, prove that the mapping $f: G \to G$ defined by $f(x) = x^m, x \in G$ is an automorphism.

2. **Answer any TWO questions** **$2 \times 5 = 10$**

(a) Let G be a group and Z(G) be the centre of G. If $G/Z(G)$ is cyclic then show that G is a commutative group If G be a finite non commutative group then show that $|Z(G)| \leq \frac{1}{4}|G|$

3+2

(b) Show that any group of order less than 6 is commutative. 5

(c) Let H, K be finite cyclic groups. Then show that $H \times K$ is cyclic if and only if $O(H)$ and $O(K)$ are relatively prime.

3. **Answer any ONE question** **$10 \times 1 = 10$**

(a) (i) State and prove Cayley's theorem for group.

(ii) Define inner automorphism of a group. Show that $G/Z(G) \cong Inn(G)$. 2+3

(b) (i) Define commutator subgroup . Suppose C is the commutator subgroup of a group G. Show that C is normal subgroup in G. Also show that the quotient group $\frac{G}{C}$ is

Abelian. Then show that if N be a normal subgroup of G, then $\frac{G}{N}$ is Abelian if and only if $C \subset N$. 1+1+1+3

(ii) Prove that a finite group of order n is isomorphic to a subgroup of S_n. Find the permutation group isomorphic to the group $G = (\{1, i, -1, -i\}, .)$. 3+1

Group-B

[Linear Algebra-II]

4. **Answer any eight questions** **$2 \times 8 = 16$**

a) Show that the solutions of the differential equation $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$ is subspace of a vector space of all real valued continuous functions.

b) Consider the following subspace of $\mathbb{R}^3$ such that $W = \{(x,y,z) \in \mathbb{R}^3 | 2x + 2y + z = 0, 3x + 3y - 2z = 0, x + y - 3z = 0\}$ .Then what is the dimension of W?

c) Let $= \{(-1,0,1), (2,1,4)\}$. For what value of k for which the vector $(3k + 2,3,10)$ belongs to $L(S)$?

d) Examine that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is linearly independent in $\mathbb{R}^3$.

e) Give an example to show that union of two vector subspace of a vector space V may not be subspace of V.

f) Define null space and range space of a linear mapping.

g) Find two linear operators T_1 and T_2 on $\mathbb{R}^2$ such that $T_1T_2 = 0$ bnt $T_2T_1 \neq 0$.

h) Let V be a vector space over a field F. If $\alpha \in F$ and $x \in V$ such that $\alpha x = x$ then show that $\alpha = 1$ or $x = 0$

i) Find the dimension of the vector space Cover $\mathbb{R}$.

j) What do you mean by quotient space of a vector space?

k) Define image and kernel of a linear mapping.

l) Let the matrix of the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to the ordered basis $B = \{(1,1),(-1,1)\}$ be $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$. Find T.

5. Answer any two questions **$5 \times 2 = 10$**

a) Show that every finite dimensional real vector space of dimension n is isomorphic to $\mathbb{R}^n$

b) Let V be a vector space of all 2×2 matrices over $\mathbb{R}$. Determine whether $A, B, C \in V$ are dependent, given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}$

c) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that, $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$. Find $T(x, y)$.

6. Answer any one question **$10 \times 1 = 10$**

a) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by $T(x, y, z) = (3x, x - y, 2x + y + z)$ for all $(x, y, z) \in \mathbb{R}$. Then check whether T is invertible? If so find T^{-1}. 5+5

b) (i) State and prove first isomorphism theorem for vector space. 2+4

(ii) State rank –nullity theorem. By using first isomorphism theorem prove the rank-nullity theorem. 2+2