# 2022

# Mathematics

## [Honours]

## (B.Sc. Third Semester End Examination-2022)

### PAPER-MTMH C303

### [Real Analysis II]

***Full Marks: 60*** ***Time: 03 Hrs***

*The figures in the right hand margin indicate marks*

*Candidates are required to give their answers in their own words as far as practicable*

*Illustrate the answers wherever necessary*

### Group-A

### [Real Analysis II]

**1. Answer any ten questions** $10 \times 2 = 20$

a. Let $f:[a,b] \to \mathbb{R}$ be continuous and strictly monotone in $[a,b]$ Let $f(a) \neq f(b)$ Let $K$ be any number between $f(a)$ and $f(b)$ Then exists exactly one point $c \in (a,b)$ such that $f(c) = K$ [using Intermediate value theorem]

b. Show that $x.2^x = 1$ has a solution in $(0,1)$

c. Prove that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$

d. Let $f:\mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and $f(q) = 0$ for every rational number $q$. Prove that $f(x) = 0 \ \forall x \in \mathbb{R}$

e. Let $f(x)=\begin{cases}0 & -1\le x\le 0\\ 1 & 0<x\le 1\end{cases}$

Is there a function F such that $P'(x)=f(x)$ in [-1,1]

f. A function f is differentiable on [0,2] and $f(0)=0,\ f(1)=2,\ f(2)=1$ Prove that $f'(c)=0$ for some $c\in(0,2)$

g. If $f'(x)$ exists and is bounded in some interval I then f is uniformly continuous on I

h. Show that there is no real number k for which the equation $x^3-3x+k=0$ has two distinct roots in (0, 1)

i. Prove that $Cosx<1-\frac{x^2}{2}+\frac{x^4}{24}$ if $0<x<\frac{\pi}{2}$

j. Write the sufficient condition for differentiability of the function $f(x,y)$ at $(a,b)$

Give an example to show that the condition is not necessary.

k. If $u=\tan^{-1}\left(\frac{x^{5/2}+y^{5/2}}{\sqrt{x}-\sqrt{y}}\right)$ show that $x\frac{\partial y}{\partial x}+y\frac{\partial y}{\partial x}=\sin 2u$

l. Let. $f(x)=Sgn\,x,\ g(x)=x\left(1-x^2\right)$ Show that the composite function g o f is continuous at 0

m. Find the limit $\lim_{x\to 0}\frac{1}{x^2}$



n. State the nature of discontinuity of $f(x)$ at $x=0$ where

$$f(x)=Sin\frac{1}{x}\quad x\neq 0$$
$$=0,\qquad x=0$$

o. Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both continuous on $[a,b]$ and both differentiable on $(a,b)$ Is $f'(x)=g'(x)\forall x\in(a,b)$ implies $f(x)=g(x)$? justify.

**2. Answer any four questions** $4\times5=20$

a. Let $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)$ is periodic with periodicity $p$ i.e there exists $p$>0 such that $f(x+p)=f(x)$ for all $x$ Prove that if $f$ is continuous on $\mathbb{R}$ then $f$ is bounded and uniformly continuous on $\mathbb{R}$

b. State and prove Role's theorem

c. Let f be a real-valued function which is continuous in the interval [a, b] and which has first and second order derivatives on $(a,b)$. If $F(x)=f(x)-\alpha-\beta x-\gamma x^2$ where $\alpha,\beta,\gamma$ are chosen so that $F(a)=F(b)=F(c)=0$

Prove that there exists $\xi\in(a,b)$ such that

$$\frac{1}{2}f''(\xi)=\begin{vmatrix}1 & a & f(a)\\ 1 & b & f(b)\\ 1 & c & f(c)\end{vmatrix}\Bigg/\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$ 5

d. If $f(x) = Cos x$ .Prove that $\lim_{h \to 0} \theta = \frac{1}{2}$ where $\theta$ is given by

$f(h) = f(0) + hf'(\theta h),\ 0 < \theta < 1$ 5

e. Find the maximum value of $8xyz$ subject to the condition

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 5

f. i) If $\phi(v^2 - x^2, v^2 - y^2, v^2 - z^2) = 0$ where $v$ is a function of $x, y, z$ show that $\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$

ii) If $v = z \tan^{-1}\frac{y}{x}$ Prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ 3+2

**3. Answer any two questions** $2 \times 10 = 20$

a. (i) Let $f : [a, b] \to \mathbb{R}$ and continuous on closed and bounded interval $[a, b]$. If $f(a)\,f(b) < 0$ then prove that there exists at least a point $c \in (a, b)$ such that $f(c) = 0$. Hence prove that if $f$ is montonic in $[a, b]$ then there exists a unique $c \in (a, b)$ such that $f(c) = 0$

ii) A function f is defined on (-1, 1) by $f(x) = x^{\alpha} \sin\frac{1}{x}, \quad x \neq 0$
$= 0 \quad x = 0$

Prove that i) if $0 < \alpha < 1$ then $f'(0)$ does not exist.



(ii) If $\alpha > 1$ then $f'(x)$ is continuous and $f'(0) = 0$. 4+2+4

b. (i) Let a function $f : \mathbb{R} \to \mathbb{R}$ Prove that $f$ is continuous on $\mathbb{R}$ if and only if $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

ii) Evaluate the limits $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ Hence deduce that $\lim_{x \to 0} f(x)$ exists or not where $f(x) = \frac{1}{e^{\frac{1}{x}} + 1}$

iii) Justify Rolle's theorem is applicable or not $f(x) = 1 - (x-1)^{\frac{1}{2}}$ on [-1, 1] 5+3+2

c. (i) Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotonically increasing function on I. Prove that

1) If $f$ is bounded above on I then $\lim_{x \to b} f(x) = \sup_{x \in (a,b)} f(x)$

2) If $f$ is bounded below on I then $\lim_{x \to a} f(x) = \inf_{x \in (a,b)} f(x)$

ii) Let $\phi : [0, 2] \to \mathbb{R}$ be defined by $\phi(x) = \lim_{n \to \infty} \frac{x^{2n+2} + \cos x}{x^{2n} + 1}$

Discuss continuity of $\phi(x)$ on [0, 2]. Hence justify the Bolzano's theorem is applicable or not on $\phi(x)$ in [0, 2]

iii) Prove that $Sin\, x < x \ \forall x \in \left(0, \frac{\pi}{2}\right)$ 4+4+2