# Mathematics

## [Honours]

## (B.Sc. Third Semester End Examination-2023)

## [Regular and Supplementary Paper]

### PAPER-MTMH C302

### (Group Theory – II and Linear Algebra-II)

***Full Marks: 60*** ***Time: 03 Hrs***

*The figures in the right hand margin indicate marks*

*Candidates are required to give their answers in their own words as far as practicable*

*Illustrate the answers wherever necessary*

**[Use separate answer script for each group]**

### Group-A

### [Group Theory-II: Marks-24]

**1. Answer any TWO questions** $2 \times 2 = 4$

a) Find the number of inner automorphisms of the group $S_3$.

b) Let $H$ be a normal subgroup of $G$ such that $|H| = 2$. Show that $H \subseteq Z(G)$ where $Z(G)$ is the centre of $G$.

c) Let $G$ be a group of order 8 and $x$ be an element of $G$ of order 4. Then show that $x^2 \in Z(G)$.

d) Give examples of two groups $G$ and $G'$ such that $G \not\cong G'$ but Aut $(G) \cong \text{Aut}(G')$

**2. Answer any TWO questions** $2 \times 5 = 10$

a) Let $H \subset K \subset G$ and $H$ is normal in $K$, $K$ is normal in $G$ and also $H$ is normal in $G$. Then show that $K/H$ is normal in $G/H$ and $\frac{G/H}{K/H} \simeq G/K$.

b) Define characteristics subgroup of a group. Prove that every characteristics subgroup of a group G is a normal subgroup of G. Is the converse true? Justify.

c) Find the number of distinct cyclic subgroups of order 10 in the group $Z_{50} \times Z_{25}$.

**3. Answer any ONE question** $10 \times 1 = 10$

a) (i) If $G$ be a finite abelian group and $d$ be a positive divisor of $O(G)$, then show that $G$ has a subgroup of order $d$.

(ii) Let $G_1, G_2$ be two groups and $Z(G_1), Z(G_2)$ be their respective centers. Then prove that $Z(G_1) \times Z(G_2)$ is the center of the group $G_1 \times G_2$.5+5

b) (i) Let $G$ be a cyclic group of order 60. Find the number of normal subgroups of $G$. Let $H$ and $K$ be two normal subgroups of a group $G$ such that $H \cap K = \{e_G\}$. Show that $ab = ba$ for all $a \in H$ and $b \in K$ 2+3



(ii) Define quotient group. Let H be a subgroup of a commutative group G.Show that G/H is a commutative group. Does the converse of this result hold? Justify. 2+1+2

## Group-B

## [Linear Algebra-II: Marks - 36]

**1. Answer any EIGHT questions** $2 \times 8 = 16$

(a) The set of all continuous function $f(x)$ on $[a, b]$ such that $f\left(\frac{a+b}{2}\right) = 1$ with usual addition of function and multiplication by real numbers is a vector space or not?

(b) Find the dimension of the subspace $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$ of $\mathbb{R}^3$.

(c) Can you construct a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $R(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3: x_1 + x_2 + x_3 = 0\}$. State reason.

(d) Check the linearly independence of $x^2$ and $x|x|$.

(e) Is $L\{1, \sqrt{2}\}$ forms a subspace of R over R, explain.

(f) Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping defined by $f(x, y) = (x + y, x - y)$ for all $(x, y) \in \mathbb{R}^2$. Is $f$ bijective? Justify.

(g) Give an example of a linear operator T on $\mathbb{R}^2$ such that $T \neq 0$ but $T^2 = 0$

(h) For what value of $\lambda$ the vectors $(\lambda, 1), (1, \lambda)$ are linearly independent in $R^2$ over R.

(i) Let $T: V \to V$ be a linear mapping such that dim $V = d$, rank of $T = r$ and nullity $T = n$. Show that $nr \le \frac{1}{4}d^2$.

(j) Prove that collection of all polynomial in $P_2$ that are divisible are $x - 2$ forms a subspace in $P_2$(all polynomial of degree $\le 2$)

(k) Find the basis of the solution space of $\frac{d^2y}{dx^2} + 4y = 0$.

(l) Show that $\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of $\mathbb{R}^3$

**2. Answer any TWO questions** $5 \times 2 = 10$

(a) Let $T: V \to V$ be a linear map such that $R(T) = Ker(T)$. Find $T^2$.

(b) State and prove deletion theorem.

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be a linear mapping and $g: \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by $g(x,y) = (x, y - f(x))$ for all $(x,y) \in \mathbb{R}^2$

Show that g is an isomorphism.

**3. Answer any ONE question** $10 \times 1 = 10$

(a) i) Let $T: V \to W$ be a linear transformation. Then if $\{u_1, u_2, \dots, u_n\}$ are in V such that $\{T(u_1), T(u_2), \dots, T(u_n)\}$ are linearly independent in $W$. Then show that $\{u_1, u_2, \dots, u_n\}$ are



linearly independent in $V$. Whether the converse is true or not, if yes prove it, if not state an example.

ii) Let $(a,b,c),(d,e,f),(g,h,i) \in \mathbb{R}^2$, then show that these vector are linearly independent iff there exists a vector $X = (x,y,z) \in \mathbb{R}^3$ such that $AX = 0$, where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

5+5

(b). i) Let $V$ and $V'$ be two finite dimensional vector spaces over the same field $F$. Show that $V$ is isomorphic to $V'$ if and only if dim $V$ = dim $V'$

ii) Let $T: V \to V$ be a linear mapping satisfying $T - T^2 = id_T$ (identify map). Show that T is invertible. 5+5