Total Pages-03 RNLKWC/P.G./IS/MTM-101/23

M.Sc. First Semester End Examination, 2023

(Regular & Supplementary Paper)

Applied Mathematics with Oceanology and Computer Programming

PAPER-MTM-101

[REAL ANALYSIS]

Full Marks: 50 ***Time: 02 Hrs***

The figures in the right hand margin indicate mark.

Candidates are required to give their answers in their own words as far as practicabl.

Illustrate the answers wherever necessary

Answer question no. 1 and any four from the rest

1. Answer any four questions: 4x2=8

a) Let X be a measurable space and $\chi_E : X \to \mathbb{R}$ be a measurable function where $\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \not\in E \end{cases}$. Is E a measurable set in X?

b) Define open cover of a metric space. Give an open cover for (0, 1).

c) Evaluate $\int_1^4 (x-[x]) \quad d(x^2)$

d) Show that the set of rational numbers is a null subset of $\mathbb{R}$.

(e) Prove that outer measure m*(A) of a set A is translation invariant.

f) Define Cantor set.

2. a) Let (X, d) be a metric space and $A \subseteq X$. If there exists a sequence of distinct points of A converging to P then prove that $p \in A'$

(b) Let X be the set of all continuous real valued function defined on [0,1] and let $d(x, y) = \int_0^1 |x(t) - y(t)|\, dt \ \forall x, y \in X$.

Check whether (X, d) is complete or not. 4+4

3. (a) Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on [a ,b] and $c \in (a,b)$ then prove that

(i) f is bounded variation on [a,c] and [c.b]

(ii) $V_f[a,b]=V_f[a,c]+V_f[c,b]$

b) Show that the function $f(x)$ defined on [2, 5] by

$$f(x) = \begin{cases} 3, \textit{for all rationals } x \textit{ in} [2,5] \\ 4, \textit{for all rationals } x \textit{ in} [2,5] \end{cases}$$ 4+4

is not a function of bounded variation [2, 5].

4. (a) If f is continuous on [a,b] and α is monotonic increasing function on [a, b] then prove that $\exists$ a point $\xi \in [a,b]$ *such that* $\int_a^b f\, d\alpha = f(\xi)[\alpha(b) - \alpha(a)]$

b) Let f(x) be defined as $f(x) = \frac{1}{x^{\frac{4}{5}}}$ if $0 < x \le 1$ and $f(0) = 0$.

Show that f is Lebesgue integrable on [0, 1]. 4+4

5. a) Suppose f is continuous on [a, b] and α is monotonically increasing on [a, b]. Show that $f \in \mathfrak{R}(\alpha)$ on [a, b].

b) If $f \in R(\alpha_1)$ *and* $f \in R(\alpha_2)$ then show that $f \in R(\alpha_1 + \alpha_2)$ and hence show that $\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$ 3+5

6. a) Let f(x) be bounded and Lebesgue integrable function on [a,b] and g(x) be a bounded function on [a,b] such that f(x)=g(x) a.e. on [a,b]. Prove that g(x) is Lebusgue integrable on [a, b] and $\int_a^b g(x)dx = \int_a^b f(x)dx$.

b) State and prove the Fatou's lemma. Give an example to show that strict inequality can occur in Fatou's lemma. 4+4

7. (a) Every bounded Riemann integrable function over [a,b] is Lebesgue integrable and the two integrals are same. Is converse true ? Justify.

b) If f(x)=0, for every x in the cantor set P_0 and f(x)=k for each x in each of the intervals of length $1/3^k$ in G_0, where G_0 is the union of deleted intervals, prove that f is Lebesgue integrable on [0,1] and that $\int_0^1 f = 3$ 4+4

[Internal Marks – 10]
