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**APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING**

# M.Sc. Fourth Semester End Examination-2024

**[Regular & Supplementary Paper]**

**PAPER-MTM-401**

**[Functional Analysis]**

***Full Marks: 50*** ***Time: 02 Hrs***

*The figures in the right hand margin indicate marks*

*Candidates are required to give their answers in their own words as far as practicable*

*Illustrate the answers wherever necessary*

**1. Answer any four questions of the following: 4x2= 8**

a) In any inner product space X, prove that if the closed unit ball is compact than X is finite dimensional.

b) Show that $(T_1T_2)^* = T_2^*T_1^*$.

c) Show that in a Hilbert space strong convergence of a sequence implies weak convergence.

d) Prove that Apollonius identity

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2.$$

e) Let $H$ be a Hilbert space and fix $y\in H$. Define $f(x)=<x,y>$ for all $x\in H$. Find $\|f\|$.

f) State the Riesz representation theorem.

**2. Answer any four questions of the following: 4x4= 16**

a) Let E be a measurable subset of $\mathbb{R}$ and for $t \in E$, let $x_1(t) = t$. Let $X = \{x \in L^2(E) : x_1 x \in L^2(E)\}$ and $F : X \to L^2(E)$ be defined by $F(x) = x_1 x$ . Show that if $E = [a,b]$, then F is continuous , but if $E = \mathbb{R}$, then F is not continuous.

b) In a real Hilbert space if $\|x\| = \|y\|$ Show that $(x+y, x-y) = 0$

Interpret the result geometrically if H is Euclidean 2-space $R^2$.

c) Let X be a normed space and $X_0$ be a closed subspace of X If $x \in X$ , then x does not belong to $X_0$ if and only if there exist a non-zero linear functional $\phi \in X^*$ such that $\phi(x) \neq 0$ and $\phi(y) = 0$ for all $y \in X_0$.

d) If T is a bounded linear operator from Hilbert space X to itself satisfying $(Tx, x) = 0$ for all x is X then show that T is a zero operator.

e) Let $T : H \to H$ be self adjoint operator. Show that all eigen values of T are real and eigen vactors corresponding to different eigen values of T are orthogonal

f) Let H be a Hilbert space and $E \subset H$ . Prove that $\overline{span(E)} = E^{\perp\perp}$



**Answer any two questions of the following: 2x8=16**

**3.** a) Let Y be a subspace of X and $g : Y \to \mathbb{C}$ be a continuous linear functional. Show that the set of all Hahn-Banach extensions of g to X is a nonempty, convex, closed and bounded subset of $X^*$.

b) Let $X = C^1[0,1]$ be equipped with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $Y = C[0,1]$ be equipped with a supremum norm $\|x\|_\infty$. Check whether the linear operator $F : X \to Y$. defined by $F(x) = x$ continuous. 5+3

**4.** a) If $T : R^2 \to \mathbb{R}^2$ defined by

$T(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Show that T is bounded linear transformation and $\|T\| = 1$

b) Let Z be a fixed member in a Hilbert space H. Show that $f(x) = (x, z)$ for all $x \in H$ is a bounded linear functional over H with $\|f\| = \|z\|$. 4+4

**5.** a) State the open mapping theorem. Explain why $Tx = \sin x + \cos x$ is not an open mapping from $\mathbb{R}$ to $\mathbb{R}$.

b) If parallelogram holds in a Banach space. Show that it is a Hilbert space. 4+4

6. a) Let $T: l^2 \to l^2$ be given by

$$T(x_1, x_2, \ldots, x_n, \ldots) = \left(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots\right).$$

Is T bounded? Is the range of T complete (as a subspace of $l^2$)?

b) Let X be A normed space and $f: x \to \mathbb{C}$ be linear. If f is discontinuous, show that $f(B_1) = \mathbb{C}$, where $B_1$ denotes the open ball centred at 0. 5+3

**[Internal Assessment – 10]**