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Inventory model with stochastic lead-time and price dependent demand incorporating advance payment

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ABSTRACT

Inventory model for an item is developed in stochastic environment with price-dependent demand over a finite time horizon. Here, probabilistic lead-time is considered and shortages are allowed (if occurs). In any business, placement of an order is normally connected with the advance payment (AP). Again, depending upon the amount of AP, unit price is quoted, i.e., price discount is allowed. Till now, this realistic factor is overlooked by the researchers. In this model, unit price is inversely related with the AP amount. Against this financial benefit, the management has to incur an expenditure paying interest against AP. Taking these into account, mathematical expression is derived for the expected average profit of the system. A closed form solution to maximize the expected average profit is obtained when demand is constant. In other cases model is solved using generalized reduced gradient (GRG) technique and stochastic search genetic algorithm (GA). Moreover, results of the models without and with advance payment are presented and solved. The numerical examples are presented to illustrate the model and the results for two models obtained from two methods are compared in different cases. Also, some parametric studies and sensitivity analyses have been carried out to illustrate the behavior of the proposed model. It is observed that advance payment has positive effect on the system.

1. Introduction

It is normally observed that after receiving an order, supplier needs some time to deliver the item, which is known as lead-time. Lead-time is normally fuzzy or stochastic in nature. A number of research papers have been already published in this direction [1,2]. Recently, Kalpakam and Sapan [3] studied a perishable inventory model with stochastic lead-time. Again in the classical inventory models, either in deterministic or probabilistic model, it is often assumed that payment will be made to the supplier for goods immediately after receiving the consignment. However, one can easily observe that a supplier provides a credit period for a retailer to stimulate the demand, to boost market share or to decrease inventories of certain items. Goyal [4] first studied an EOQ model under the conditions of permissible delay in payments. Chung [5] presented the discounted cash flow (DCF) approach for the analysis of the optimal inventory policy in the presence of the trade credit. Later, Shinn et al. [6] extended Goyal's [4] model and considered quantity discounts for freight cost. Recently, to accommodate more practical features of the real-life inventory systems, Aggarwal and Jaggi [7] and Hwang and Shinn [8] extended Goyal's model to consider the deterministic inventory model with a constant deterioration rate. Shah and Shah

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[9] developed a probabilistic inventory model when delay in payments is permissible. They developed an EOQ model for deteriorating items in which time and deterioration of units are treated as continuous variable and demand is a random variable. Later on, Jamal et al. [10] extended Aggarwal and Jaggi's [7] model to allow for shortages and make it more applicable in real world.

All the above inventory models overlooked the situation of advanced payment of ordering an item. It is normally observed that a wholesaler demands some payment when an order from a retailer is placed. Also in some situations it is observed that if a retailer gives an extra advance-payment (AP) then he/she may get some price discount at the time of final payment (e.g., bricks and tiles factories announce such offer at the beginning of the season). Though AP is a real life phenomenon, it has not been addressed till now. Here an attempt has been made to incorporate this phenomenon where lead-time for delivery of an item is stochastically governed by some feasible distribution.

In the present competitive market, the selling price of a product is one of the decisive factors in selecting the item for use. In practice, higher selling price of a product negates the demand where as reasonable and low price has a reverse effect. This argument is more appropriate for defective goods whose demand is always price dependent. Whitin [11] first presented an inventory model considering the effect of price dependent demand. Generally this type of demand is seen in finished goods. Though extensive research work has been done in this area [12–15], very few of them have considered selling price as a decision variable [16,17].

Taking the above shortcomings into account, here an inventory model is developed where demand of the item depends on selling price, lead-time is stochastic in nature, retailer has to pay some advance payment at the time of ordering and is eligible for a price discount against extra advance payment. Shortages are completely backlogged and are met as soon as new order arrives. Here objective is to maximize the expected average profit. A GA, based on Roulette selection, whole arithmetic crossover and non-uniform mutation is developed for the model and objective function is optimized using it. Results via GA are compared with a non-linear optimization technique, GRG, in some particular cases. In a particular situation when demand is constant, closed form solution is obtained. Models are illustrated with some numerical examples and some sensitivity analyses have been presented. Results for the models with and without advance payment are obtained and it is observed that profit is more when advance payment is allowed and price discount due to that is permitted.

2. Assumptions and notations for the proposed models

Single-item inventory model with shortages in a finite time horizon is developed using the following notations and assumptions:

X	lead-time which is a random variable (r.v)
x	real variable corresponding to r.v. X
$f(x)$	density function of X
$F(x)$	distribution function of X
m	mean of X when it follows normal distribution
σ	standard deviation of X when it follows normal distribution, and the value of σ is sufficiently small so that probability of $X \leqslant 0$ is negligible (see Section 5)
λ	parameter of exponential distribution when X follows exponential distribution
C_3	set-up cost per cycle
P	selling price per unit item
p	unit cost per unit item
s	mark-up of selling price, i.e., $P = sp$, which is a decision variable
n	total number of replenishment to be made during the prescribed time horizon
H	prescribed time horizon, where H is sufficiently larger than mean of X so that probability of $n < 2$ is negligible (see Section 5)
$D(P)$	demand per unit time is a function of selling price, i.e., $D(P) = D_0 P^{-\gamma}$
T	equal length of each time-cycle, i.e., $T = H/n$
h	holding cost per unit item per unit time
$q(t)$	inventory level at time t
Q	maximum inventory level
Q_r	re-order level when permissible AP allowed (decision variable)
A_p	advance payment for purchasing quantities
S_p	total selling price over the planning horizon H
P_c	total purchasing cost over planning horizon H
C_s	shortage cost per unit item per unit time
I_b	percentage of bank interest
I_d	percentage of AP on total purchase cost per cycle
I_{d0}	minimum percentage of AP on total purchase cost per cycle, which is mandatory
I_c	percentage of discount on unit cost, which is a function of I_d and is of the form: $I_c = k - k\left(\frac{100-I_d}{100-I_{d0}}\right)^2$, where k is a constant $(0 < k < 100)$
Q_r^*	optimum value of Q_r
Z^*	optimum value of the profit function Z

Q^* optimum value of Q
T^* optimum value of T
s^* optimum value of s

3. Model development and analysis

To develop the proposed inventory model, we assume that business starts with an inventory level of Q units of the item. At the beginning of every renewable cycle (i.e., at $t = t_j$ for jth renewable cycle, $j = 1, 2, \ldots, n-1$), when the inventory level reaches at reorder level Q_r, then new order placed to meet the customer demand for the next cycle. After delivery of the order, at first, shortages (if any) of previous cycle are fulfilled and then the rest of the order quantities are kept in store where it is used to meet the demand. In line with most researchers [18], we assume that immediately after the arrival of an order the installation stock will always exceed the reorder level, so at most one order will be outstanding at any time. When the stock level reaches to the reorder level Q_r at $t = t_{j+1}$ order for the next $(j+1)$th renewable cycle is made. Thus, in the interval $[0, H]$, including the initial order, n ordering points are at $t = 0$, $t = t_j$, $(j = 1, 2, \ldots, n-1)$. Shortages are not allowed for the last cycle. The time X between placement and receipt of an order is random in nature and is assumed to follow a feasible distribution. The pictorial representation of the inventory system is depicted in Fig. 1.

The differential equation of the inventory system is given by

$$\frac{dq(t)}{dt} = -D(P), \tag{1}$$

with the initial conditions, $q(0) = Q = D(P)T$, $t_1 = \frac{Q-Q_r}{D(P)}$, $q(t_j) = Q_r$, $j = 1, 2, \ldots, n$. The solution of the differential Eq. (1) is

$$q(t) = \begin{cases} Q_r - D(P)(t - t_j), & t_j \leqslant t \leqslant t_j + x \\ Q_r + Q - D(P)(t - t_j), & t_j + x \leqslant t \leqslant t_{j+1} = t_j + T \end{cases} \tag{2}$$

for $j = 1, 2, \ldots, n-1$.

According to the assumptions, in each renewable cycle (t_j, t_{j+1}), two cases may arise.

Case-1: When no shortages occur.

Case-2: When shortages occur.

Calculation for jth *renewable cycle* $[t_j, t_{j+1}]$

For jth cycle, in case-1 the holding cost H1(x) is given by

$$H1(x) = h\left[\int_{t_j}^{t_j+x} q(t)\,dt + \int_{t_j+x}^{t_{j+1}} q(t)\,dt\right] = \frac{h}{2D(P)}[(Q_r + Q - D(P)x)^2 - (Q_r - D(P)x)^2]. \tag{3}$$

In case-2 the holding cost $H2(x)$ is given by

$$H2(x) = h\left[\int_{t_j}^{t_j+Q_r/D(P)} q(t)dt + \int_{t_j+x}^{t_{j+1}} q(t)dt\right] = \frac{h}{2D(P)}[(Q_r + Q - D(P)x)^2]. \tag{4}$$

So expected holding cost in jth cycle $E(H(x))$ is given by

$$E(H(x)) = \frac{h}{2D(P)}\left[\int_0^{Q_r/D(P)} \{(Q_r + Q - D(P)x)^2 - (Q_r - D(P)x)^2\} f(x)dx + \int_{Q_r/D(P)}^{\infty} \{(Q_r + Q - D(P)x)^2\} f(x)dx\right] \tag{5}$$

$$= \frac{h}{2D(P)}\left[\int_0^{Q_r/D(P)} \{(Q^2 + 2QQ_r) - 2QD(P)x\} f(x)dx + \int_{Q_r/D(P)}^{\infty} \{(Q_r + Q)^2 - 2(Q_r + Q)D(P)x - (D(P)x)^2\} f(x)dx\right]$$

$$= \frac{h}{2D(P)}[\{(Q^2 - 2QQ_r)F(Q_r/D(P)) + 2QD(P)I_{11}\} + \{(Q_r + Q)^2(1 - F(Q_r/D(P))) - 2(Q_r + Q)D(P)I_{21} - D(P)^2 I_{22}\}], \tag{6}$$

where

$$I_{11} = \int_0^{Q_r/D(P)} xf(x)dx, \tag{7}$$

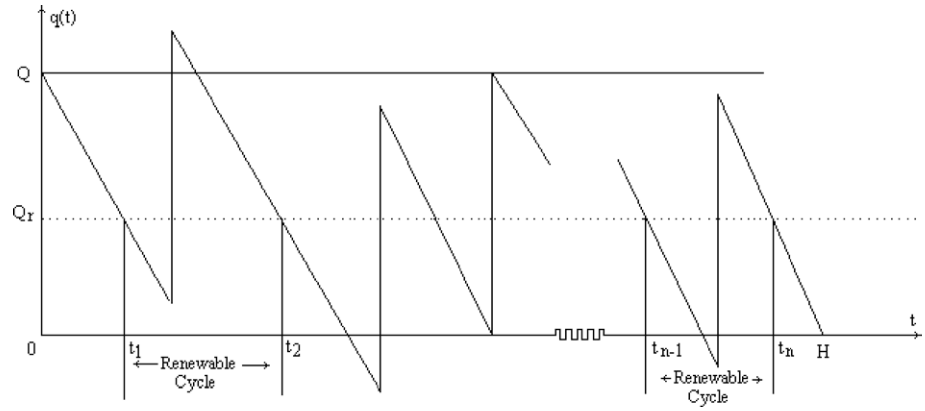

Fig. 1. Instantaneous state of inventory system.

$$I_{21} = \int_{Q_r/D(P)}^{\infty} x f(x) \mathrm{d}x, \tag{8}$$

$$I_{22} = \int_{Q_r/D(P)}^{\infty} x^2 f(x) \mathrm{d}x \tag{9}$$

and expressions of $F(Q_r/D(P))$, I_{11}, I_{21} and I_{22} for normal and exponential distributions of X are given in Appendix 1.

The shortage cost $S_c(x)$ is given by

$$S_c(x) = C_s \int_{t_j+Q_r/D(P)}^{t_j+x} -q(t)\mathrm{d}t = \frac{C_s\{Q_r - D(P)x\}^2}{2D(P)}. \tag{10}$$

So expected shortage cost $ES_c(x)$ is given by

$$E(S_c(x)) = \frac{C_s}{2D(P)} \int_{Q_r/D(P)}^{\infty} \{Q_r - D(P)x\}^2 f(x)\mathrm{d}x = \frac{C_s}{2D(P)}\{Q_r^2(1 - F(Q_r/D(P))) - 2Q_r D(P) I_{21} + D(P)^2 I_{22}\} \tag{11}$$

Calculation of expected average profit in H

Holding cost for the time period $[0, t_1]$ and $[t_n, H]$, $H3$, is given by

$$H3 = h\int_0^{Q/D(P)} q(t)\mathrm{d}t = \frac{hQ^2}{2D(P)}. \tag{12}$$

For each cycle, total selling price S_p, total purchase cost P_c, advance payment for purchasing quantities A_p, interest on loan from bank L_p, expected interest on loan from bank $E(L_p)$ are given respectively by $S_p = QP$, $P_c = Qp$, $A_p = I_d Qp(1 - I_c)$, $L_p = A_p X I_b$ and $E(Lp) = A_p m I_b$, $m = E(X)$ being the mean of the distribution of X.

So total expected profit over the planning horizon = ⟨Sales revenue⟩ − ⟨Purchase cost⟩ − ⟨Expected interest⟩ − ⟨Ordering cost⟩ − ⟨Expected holding cost⟩ − ⟨Expected shortage cost⟩.

Hence, the expected average profit during the planning horizon H is

$$Z = [nS_p - nP_c - (n-1)E(L_p) - nC_3 - (n-1)E(H(x)) - H3 - (n-1)E(S_c(x))]/H. \tag{13}$$

So the problem reduces to determine Q_r and s to maximize Z. GRG technique is used to find local optima of the objective function Z and a heuristic method GA is also used to find the solution. As GA searches for a global optima for an objective function in a search space it is used to compare the result obtained via GRG.

3.1. Particular case

Here, demand is assumed as constant, i.e., $\gamma = 0$

In this particular case $D(P) = D_0$ and so Z is a function of Q_r only and can be optimized analytically.

Theorem 1 (Differentiation under the integral sign). *If a function $f(x,y)$ is defined and continuous on the rectangle $R = [a, b : c, d]$ and if*

(i) $f_x(x,y)$ exists and is continuous on the rectangle R
(ii) $\phi : [a,b] \to [c,d]$ and $\psi : [a,b] \to [c,d]$ are both differentiable
(iii) $g(x) = \int_{\phi(x)}^{\psi(x)} f(x,y)dy$ for $x \in [a,b]$ then $g(x)$ is differentiable on $[a,b]$, and

$$g'(x) = \int_{\phi(x)}^{\psi(x)} f_x(x,y)dy + f(x,\psi(x))\psi'(x) - f(x,\phi(x))\phi'(x)$$

for $x \in [a,b]$

Proof. The proof of this theorem is available in standard texts (cf. Apostol [19]). □

Proposition 1. *For* $\gamma = 0$, *the expected shortage cost* $E(S_c(x))$ *is convex in* Q_r.

Proof. For $\gamma = 0$, the demand $D(P)$ becomes D_0. The expected shortage cost is a function of only one variable Q_r. Now, differentiating (11) with respect to Q_r using Theorem 1, we have

$$\frac{dE(S_c(x))}{dQ_r} = \frac{C_s}{D_0}\int_{Q_r/D_0}^{\infty}\{Q_r - D_0x\}f(x)dx = \frac{C_s}{D_0}[Q_r(1 - F(Q_r/D_0)) - D_0I_{21}], \tag{14}$$

expressions of I_{21} can be obtained from (8) by replacing $D(P)$ by D_0.

Thus, the optimal value of Q_r, Q_r^*, is the solution of $\frac{dE(S_c(x))}{dQ_r} = 0$, i.e., the solution of

$$Q_r(1 - F(Q_r/D_0)) - D_0I_{21} = 0. \tag{15}$$

The condition for optimality is

$$\frac{d^2E(S_c(x))}{dQ_r^2} = \frac{C_s}{D_0}\int_{Q_r/D_0}^{\infty} f(x)dx \quad [\text{using Theorem-1}] = \frac{C_s}{D_0}[1 - F(Q_r/D_0)] > 0 \text{ for } Q_r = Q_r^*. \tag{16}$$

Hence, expected shortage cost $E(S_c(x))$ is convex in Q_r. □

Proposition 2. *For* $\gamma = 0$, *the expected holding cost* $E(H(x))$ *is convex in* Q_r.

Proof. For $\gamma = 0$, the demand $D(P)$ becomes D_0. The expected holding cost is a function of only one variable Q_r. Now, differentiating (6) with respect to Q_r using Theorem 1, we have

$$\begin{aligned}\frac{dE(H(x))}{dQ_r} &= \frac{h}{D_0}\left[\int_0^{\infty}\{Q_r + Q - D_0x\}f(x)dx - \int_0^{Q_r/D_0}\{Q_r - D_0x\}f(x)dx\right]\\ &= \frac{h}{D_0}\left[\{Q_r + Q - D_0E(X)\} - \left\{Q_rF(Q_r/D_0) - D_0\int_0^{Q_r/D_0} xf(x)dx\right\}\right]\\ &= \frac{h}{D_0}[\{Q_r + Q - D_0E(X)\} - \{Q_rF(Q_r/D_0) - D_0(1 - I_{11})\}],\end{aligned} \tag{17}$$

expressions of I_{11} can be obtained from (7) by replacing $D(P)$ by D_0.

Thus, the optimal value of Q_r, Q_r^*, is the solution of $\frac{dE(H(x))}{dQ_r} = 0$, i.e., the solution of

$$\{Q_r + Q - D_0E(X)\} - \{Q_rF(Q_r/D_0) - D_0(1 - I_{11})\} = 0. \tag{18}$$

The condition for optimality is

$$\frac{d^2E(H(x))}{dQ_r^2} = \frac{h}{D_0}[1 - F(Q_r/D_0)] > 0 \text{ for } Q_r = Q_r^*. \tag{19}$$

Hence, expected shortage cost $E(H(x))$ is convex in Q_r. □

Proposition 3. *For* $\gamma = 0$, *the expected average profit* $Z(Q_r)$ *is concave in* Q_r.

Proof. From Eq. (13), we get the expected average profit function in Q_r. Differentiating (13) with respect to the decision variable Q_r, we have

$$\frac{dZ}{dQ_r} = -(n-1)\left[\frac{DE(H(x))}{dQ_r} + \frac{dES_c(x)}{dQ_r}\right]. \tag{20}$$

For maximum value Z, $\frac{dZ}{dQ_r} = 0$, i.e., the solution of Eqs. (15) and (18) will give value of Q_r to maximize Z and let this value be Q_r^*. The condition for optimality,

$$\frac{d^2Z}{dQ_r^2} = -(n-1)\left[\frac{d^2E(H(x))}{dQ_r^2} + \frac{d^2ES_c(x)}{dQ_r^2}\right] < 0 \tag{21}$$

$$\text{when } Q_r = Q_r^* \text{ and } n > 1, \quad [\text{using (16) and (19)}]$$

Hence, due to convexity of expected shortage and holding costs, the expected profit function is concave in the decision variable Q_r. Thus, we get the closed form solution from the expressions (15) and (18). □

4. Genetic algorithm

Genetic algorithms are heuristic search process for optimization that resembles natural selection and have been developed by Holland [20], his colleagues and students at the University of Michigan (c.f. Goldberg [21]). According to Goldberg [21], Davis [22], Michalewicz [23] Genetic Algorithms are adaptive computational procedures which are modelled as the mechanics of natural genetic systems. Because of its generality and other advantages over conventional optimization methods it has been successfully applied to different decision making problems in different areas like travelling salesman problems (Forrest [24]), Scheduling problem (Davis [25]), Numerical optimization (Michalewicz [23]), etc. But, till now, only a very few researchers have applied it in the field of Inventory control system. Among them, one may refer to the works of Sarkar and Newton [26] and Mondal and Maiti [27]. In most cases, they can find the global optimum solution with a high probability. They mimic the process of natural selection and is based on Darwin's survival of the fittest principles. In this algorithm, a population of individuals (potential solutions) undergoes a sequence of unary (mutation type) and higher order (crossover type) transformations. These individuals select the next generation. This new generation contains a higher proportion of the characteristics possessed by the'good' members of the previous generation and in this way good characteristics are spread over the population and mixed with other good characteristics. After a few number of generations, the program either converges or is terminated and the best individual is taken as the optimal solution. It is generally accepted that any Genetic Algorithm to solve a problem must have the following basic components:

- Values of parameters (population size, probabilities of applying genetic operators, etc.) of Genetic Algorithms.
- Chromosome representation.
- Initial population production.
- Evaluation function rating solutions in terms of their fitness.
- Selection process
- Genetic operators (crossover and mutation) that alter the genetic composition of parents during reproduction.

Here, GA is used to verify the results obtained by GRG method. The stepwise procedure of Genetic Algorithm is shown as below:

GA Algorithm

1. Begin
2. Initialize maximum generation number (M), population size (N), probability of crossover (p_c) & mutation(p_m).
3. $t \leftarrow o$ [t represents the number of current generation]
4. Initialize $p(t)$ [$p(t)$ represents the population at t-th generation].
5. Evaluate($p(t)$). [This function is assigned fitness to each solution.]
6. While ($t < M$)
    a. $t \leftarrow t + 1$.
    b. Select $p(t)$ from $p(t-1)$.
    c. Alter (crossover and mutate) $p(t)$.
    d. Evaluate ($p(t)$).
7. End While
8. Print the best result
9. End

4.1. GA procedures for the proposed model

The different steps of the proposed genetic algorithm to find optimal decision of the model are discussed below:

(a) *Representation:* A two-dimensional real vector $Q_i = (q_{i1}, q_{i2})$ is used to represent i-th solution, where q_{i1} and q_{i2} represent two decision variables Q_r and s, respectively of the problem.
(b) *Initialization:* $N(= 20)$ such solutions $Q_1, Q_2, \ldots Q_N$ are randomly generated by random number generator such that each solution satisfies the resource constraints of the problem. The constraints are checked using a separate subfunction named check_constraint(). Also set $M = 500$, $p_c = 0.6$, $p_m = 0.1$. These values of p_c and p_m give better result for the proposed model.
(c) *Fitness value:* The value of the objective function $Z(Q_i)$ due to the potential solution $Q_i = (q_{1i}, q_{2i})$ is taken as fitness value.

(d) *Selection process for mating pool:* There are several approaches to select solutions from the initial population for mating pool [21,23]. All these approaches has some merits and demerits over the others. Among these approaches Roulette wheel selection process [23] plays a major role. In this study, Roulette wheel selection process is used. Following are the steps of this process:
   (i) Find total fitness of the population $F=\sum_{i=1}^{N} Z(Q_i)$
   (ii) Calculate the probability of selection f_i of each solution Q_i by the formula $f_i = Z(Q_i)/F$.
   (iii) Calculate the cumulative probability cp_i for each solution Q_i by the formula $cp_i = \sum_{j=1}^{i} f_i$
   (iv) Generate a random number 'r' from the range [0,1].
   (v) If $r < cp_1$ then select Q_1 otherwise select $Q_i (2 \leqslant i \leqslant N)$ where $cp_{i-1} \leqslant r < cp_i$.
   (vi) Repeat steps (iv) and (v) N times to select N solutions for mating pool. Clearly one solution may be selected more than once.
   (vii) Selected solution set is denoted by $P^1(T)$ in the proposed GA algorithm.

(e) *Crossover:*
   (i) Selection for crossover: For each solution of $P^1(T)$ generate a random number r from the range [0,1]. If $r < p_c$ then the solution is taken for crossover, where p_c is the probability of crossover.
   (ii) Crossover process: Crossover taken place on the selected solutions. For each pair of coupled solutions Y_1, Y_2 a random number c is generated from the range [0,1] and Y_1, Y_2 are replaced by their offspring's Y_{11} and Y_{21} respectively where $Y_{11} = cY_1 + (1-c)Y_2, Y_{21} = cY_2 + (1-c)Y_1$.

(f) *Mutation:*
   (i) Selection for mutation: For each solution of P^1 generate a random number r from the range [0,1]. If $r < p_m$ then the solution is taken for mutation, where p_m is the probability of mutation.
   (ii) Mutation process: To mutate a solution $Q_i = (q_{i1}, q_{i2})$ select a random integer r in the set {1,2}. Then replace q_r by randomly generated value within the boundary of r-th component of Q_i.

4.2. Implementation and usage

The algorithm is implemented using C-language to optimize the objective function of the proposed model. For each set of numerical data for the proposed model, the program is run several times and in most of the cases this GA gives almost same result. The best result found is taken as near optimum solution.

5. Numerical illustration

5.1. General inventory model with advance payment

The model is illustrated with two examples as given below.

Example 1. Here it is assumed that lead-time X is normally distributed with a known mean m and standard deviation σ. Different parametric values for this example are $C_3 = \$26$, $n = 12$, $m = 0.3$, $\sigma = 0.06$, $I_d = 30$, $I_{d0} = 20$, $I_b = 6$, $D_0 = 10,000$, $H = 8$, $p = \$4.0$, $h = \$0.6$, $C_s = \$2$, $k = 20$, $\gamma = 2.6$ and results are obtained by both GRG technique using LINGO software and GA method using C-language and presented in Table 1.

Example 2. Here it is assumed that lead-time X is exponentially distributed with a known parameter $\lambda = 3.33$. Different parametric values for this example are same as Example 1. As Example 1, results are obtained by both GRG and GA and presented in Table 1.

From the above result, it is clear that results obtained by both GA and GRG techniques are almost same. In that case also, it is observed that expected average profit for normal distribution is more than that of exponential distribution.

Table 1
Results for general inventory models with advance payment

Example	Technique	Q_r^*	s^*	$Z^*(\$)$
	GA	11.065	1.625	153.816
Example 1	GRG	11.294	1.623	153.817
	GA	14.092	1.683	143.032
Example 2	GRG	14.123	1.683	143.032
Example 3	Analytical approach	4.306	–	282.731
Example 4	Analytical approach	5.244	–	276.643

5.1.1. Particular case ($\gamma = 0$)

In this particular model demand is price independent and is solved analytically as discussed in Section 3.1. It is illustrated with two examples as given below.

Example 3. Here it is assumed that lead-time X is normally distributed with a known mean m and standard deviation σ. Different parametric values for this example are $C_3 = \$25$, $n = 14$, $m = 0.22$, $\sigma = 0.04$, $I_d = 30$, $I_{d0} = 20$, $I_b = 6$, $D = 60$, $s = 1.5$, $H = 8$, $p = \$10.3$, $h = \$0.7$, $C_s = 2$, $k = 20$.

Example 4. Here it is assumed that lead-time X is exponentially distributed with a known parameter $\lambda = 4.54$. All other parametric values are same as Example 3. Optimal decisions for Examples 3 and 4 are obtained by solving Eqs. (15) and (18) and are presented in Table 1.

5.2. Inventory models without advance payment

Examples 1–4 are solved not taking advance payment into account and the results are presented in Table 2.

From this table, it is observed that in all cases, profit is more when advance payment is made and the price discount due to it is permitted.

5.3. Sensitivity analysis

Using the numerical Example 1, mentioned earlier, the effect of under or over estimation of various parameters on re-order level, mark-up and expected average profit is studied. Here, we employ, $\Delta Q_r = (Q_r' - Q_r)/Q_r \times 100\%$,

Table 2
Results for inventory models without advance payment

Example	Technique	Q_r^*	s^*	Z^* (\$)
	GA	10.330	1.687	141.474
Example 1	GRG	10.341	1.690	141.475
	GA	12.950	1.746	131.771
Example 2	GRG	12.961	1.750	131.774
	GA	4.308	–	255.928
Example 3	GRG	4.315	–	255.929
	GA	5.240	–	249.843
Example 4	GRG	5.240	–	249.847

Table 3
Sensitivity analysis of general model for Example 1

Change of parameters		−20%	−10%	−5%	5%	10%	20%
γ	ΔQ_r	86.217	42.873	19.711	−7.455	−6.371	−6.242
	Δs	18.400	7.228	3.393	−2.799	−5.903	−9.969
	ΔZ	219.649	80.350	34.852	−26.878	−47.797	−77.178
D_0	ΔQ_r	−6.371	−6.507	−6.507	5.664	9.353	23.033
	Δs	−0.615	−0.061	−0.061	−0.117	0.184	−0.009
	ΔZ	−25.102	−12.535	−6.269	6.268	12.533	25.071

Table 4
Sensitivity analysis on mean m for Examples 1 and 3

Demand	Mean (m)	Q_r^*	s^*	Z(\$)
Example 3	0.24	5.508	–	282.534
	0.28	7.909	–	282.140
	0.32	10.308	–	281.746
	0.36	12.709	–	281.352
	0.40	14.988	–	280.958
Example 1	0.32	12.685	1.629	153.715
	0.36	16.101	1.619	153.522
	0.40	18.95	1.625	153.333
	0.44	21.865	1.624	153.138
	0.48	23.858	1.620	152.725

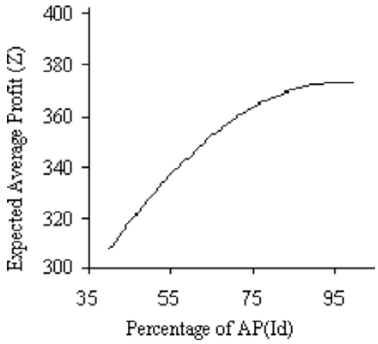

Fig. 2. I_d(%) vs. Z (demand is constant).

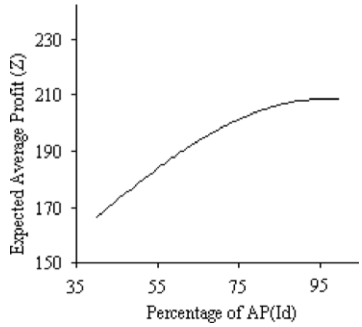

Fig. 3. I_d(%) vs. Z (demand is price dependent).

$\Delta s = (s' - s)/s \times 100\%$ and $\Delta Z = (Z' - Z)/Z \times 100\%$ as measure of sensitivity, where Q_r, s and Z are the true values and Q_r', s' and Z' the estimated values. The sensitivity analysis is shown by increasing or decreasing the parameters by 5%, 10% and 20%, taking one at a time and keeping the others at their true values. The results are presented in Table 3, which are self-explanatory.

From Table 3, it is observed that ΔZ is symmetric with respect to D_0 but it increases at a much faster rate when γ decreases than the rate of decrease when γ increases.

For numerical Examples 3 and 1, results are obtained for different mean m and results are presented in Table 4. It is observed that profit decreases as mean of the normal distribution increases for both constant (Example 3) and price dependent (Example 1) demand. It happens because as m increases, lead-time increases which increases expected holding and shortage cost which in turn decreases average profit.

Expected average profits (Z) for different percentage of advance payment ($I_d\%$) are calculated for Examples 1 and 3 and are plotted in Figs. 2 and 3, respectively. It is observed that average profit increases with I_d. But it is seen that rate of increase of profit slowly decreases with the increase of percentage of advance payment. This phenomenon agrees with reality. It is

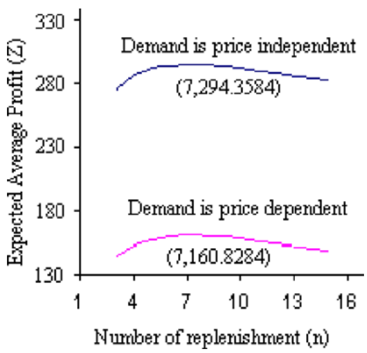

Fig. 4. Z vs. n (for normal distribution).

often observed that a wholesaler needs a large amount of investment capitals to meet all the retailers demand. In fact it is difficult for him to arrange such a large amount from bank-loan (due to lack of mortgage property). In this situation, he offers some opportunity to his retailers by giving some discount from his profit, if retailers deposit some extra amount of advance payment. Retailer takes this opportunity to make some extra profit if he has some mortgage property to take loan from bank. This phenomenon is observed in Figs. 2 and 3.

Results are obtained for different n for both the Examples 1 and 3 and are plotted in Fig. 4. It is observed in both the cases that as n increases, initially profit increases and attains maximum value ($Z = 294.3584$ at $n = 7$ for example-3 and $Z = 160.8284$ at $n = 7$ for Example 1). If n is further increases then profit gradually decreases. Since planning horizon is fixed Z will be maximum for optimum T, i.e., for optimum n. If n is below or above this optimum value then profit is less. This phenomenon is verified by this study.

6. Conclusion

A realistic inventory problem for a retailer is developed when retailer give some advance payment to wholesaler. Many research papers relating to permissible delay in payments has been published in different journals. But, till now none has considered the inventory model with advance payment incorporating stochastic lead-time. In a particular case, closed form of solution is obtained when the demand is constant. Here, benefit of the advantage of advance payment over the advance payment is illustrated. Moreover, here an algorithm in C for GA is developed and is used to verify the results obtained by GRG. GA algorithm has been implemented in C-language and executed with different seeds of random number generators. It is observed that all these executions leads to the same optimum solution. From this it may be concluded that the present solution is global optimum. These models are applicable in the factory like bricks, tiles etc. The said model can be formulated in fuzzy, fuzzy-stochastic environments. Two types of discounts can also be formulated to the proposed inventory model. Moreover, consideration of a fixed time horizon inventory model of this type will be more realistic one. All these may be the topics of future research.

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Appendix 1. Expressions of $F(Q_r/D(P))$, I_{11}, I_{21} and I_{22} for normal distribution of X is presented below.

When X is normally distributed:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}, \tag{22}$$

$$\begin{aligned} F(Q_r/D(P)) &= \int_0^{Q_r/D(P)} f(x)\mathrm{d}x = \int_{K_1}^{K_2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z = \int_{K_1}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z - \int_{K_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z \\ &= \frac{1}{1+\exp\{2A_1K_1(1+A_2K_1^2)\}} - \frac{1}{1+\exp\{2A_2K_2(1+A_2K_2^2)\}}, \end{aligned} \tag{23}$$

where

$$K_1 = -\frac{m}{\sigma}, \quad K_2 = -\frac{Q_r/D(P)-m}{\sigma}, \quad A_1 = \sqrt{\frac{2}{\pi}}, \quad A_2 = 0.044715.$$

This approximate form due to Page [28] and Tocher [29] is used to find the integrals. So

$$\begin{aligned} I_{11} &= \int_0^{Q_r/D(P)} xf(x)\mathrm{d}x = \int_{K_1}^{K_2}\frac{1}{\sqrt{2\pi}}(m+\sigma z)\exp\left\{-\frac{z^2}{2}\right\}\mathrm{d}z \\ &= m\left[\frac{1}{1+\exp\{2A_1K_1(1+A_2K_1^2)\}} - \frac{1}{1+\exp\{2A_1K_2(1+A_2K_2^2)\}}\right] - \frac{1}{\sqrt{2\pi}}\left\{\exp\left(\frac{-K_2^2}{2}\right) - \exp\left(\frac{-K_1^2}{2}\right)\right\}, \end{aligned} \tag{24}$$

$$I_{21} = \int_{Q_r/D(P)}^{\infty} xf(x)\mathrm{d}x = \int_{K_2}^{\infty}\frac{1}{\sqrt{2\pi}}(m+\sigma z)\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z = \frac{m}{1+\exp\{2A_1K_2(1+A_2K_2^2)\}} + \frac{\sigma}{\sqrt{2\pi}}\exp\left(\frac{-K_2^2}{2}\right), \tag{25}$$

$$I_{22} = \int_{Q_r/D(P)}^{\infty} x^2 f(x)\mathrm{d}x = \int_{K_2}^{\infty} \frac{1}{\sqrt{2\pi}}(m+\sigma z)^2 \exp\left(-\frac{z^2}{2}\right)\mathrm{d}z$$
$$= m^2 \int_{K_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)\mathrm{d}z + 2m\sigma \int_{K_2}^{\infty} \frac{1}{\sqrt{2\pi}} z \exp\left(-\frac{z^2}{2}\right)\mathrm{d}z + \sigma^2 \int_{K_2}^{\infty} \frac{1}{\sqrt{2\pi}} z^2 \exp\left(-\frac{z^2}{2}\right)\mathrm{d}z$$
$$= \frac{m^2+\sigma^2}{1+\exp\{2A_1K_2(1+A_2K_2^2)\}} + \frac{\sigma^2 K_2 + 2m\sigma}{\sqrt{2\pi}} \exp\left(-\frac{K_2^2}{2}\right). \quad (26)$$

Expressions of $F(Q_r/D(P))$, I_{11}, I_{21} and I_{22} for exponential distribution of X is given below.

When X is exponentially distributed:

$$f(x) = \lambda \exp(-\lambda x), \quad (27)$$

$$F(Q_r/D(P)) = \int_0^{Q_r/D(P)} f(x)\mathrm{d}x = \left(1 - \exp\left(-\frac{\lambda Q_r}{D(P)}\right)\right). \quad (28)$$

So

$$I_{11} = \int_0^{Q_r/D(P)} x f(x)\mathrm{d}x = -\frac{Q_r \exp(-\lambda Q_r/D(P))}{D(P)} - \frac{(\exp(-\lambda Q_r/D(P)) - 1)}{\lambda}, \quad (29)$$

$$I_{21} = \int_{Q_r/D(P)}^{\infty} x f(x)\mathrm{d}x = \frac{Q_r \exp(-\lambda Q_r/D(P))}{D(P)} + \frac{\exp(-\lambda Q_r/D(P))}{\lambda}, \quad (30)$$

$$I_{22} = \int_{Q_r/D(P)}^{\infty} x^2 f(x)\mathrm{d}x = \frac{Q_r^2 \exp(-\lambda Q_r/D(P))}{D^2(P)} + \frac{2I_{21}}{\lambda}. \quad (31)$$

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