APPENDIX B

# Answers to Selected Exercises

Here we provide answers to selected exercises from Chapters 1–10.

## B.1 Answers to selected exercises from Chapter 1

**Exercise 1.1.** Lewis (1986a, p. 208) assumes that all elements of a possible world are to "stand in suitable external relations, preferably spatiotemporal". Somewhat similarly, in Branching Space-Times any two point events from *Our World* are linked by appropriately combined instances of the pre-causal relation $<$ (see the $M$ property, Fact 2.4). Discuss whether the pre-causal relation (which is formally explained in Chapter 2.1) is a "suitable external relation" from Lewis's perspective.

*Answer:* It is useful to focus on the notion of "spatiotemporal relations", Lewis's paradigm for a relation to be used to draw a distinction between objects that inhabit one world, and objects that do not inhabit one world. Lewis acknowledges that there is an ambiguity in this notion. The exercise asks for an assessment from Lewis's perspective. This calls for some familiarity with his metaphysics, as we need to grasp the notion as he understood it. The central tenet of Lewis's metaphysics is the reduction of modal claims to mereological relations obtaining between certain maximal objects called "possible worlds". Thus, within a given possible world there are no (non-trivial) modal relations. On the hypothesis of so-called Humean Supervenience, a world can be identified with a "Humean mosaic", which is an ascription of properties ("perfectly natural properties") to point-like bearers. A natural option to understand such bearers is to identify them with spatio-temporal points (this appears to be Lewis's preferred option). Such points stand in various spatio-temporal relations. Paradigmatically, a spatio-temporal relation is identified with some distance between points in space-time. As to the ascription of properties, each point has exactly one set of jointly instantiable properties assigned. No property is modal, so that, for example, "possibly going up" is not an admissible property. Now, the $M$-property in the form of a real $M$ (rather than just a part of it) relates objects (idealized to be point-like) that are incompatible; such objects cannot occur together. By Lewis's central tenet, these objects cannot inhabit one and the same mosaic. If they could, they would be related by the spatio-temporal relations that underlie a given mosaic, and hence they would be compatible. And if they do not belong to one and the same mosaic, they do not belong to one and the same possible world. Thus, from Lewis's perspective, the $M$-property does not relate inhabitants of one world.

A reflection on this argument shows that it assumes that each mosaic has its own space-time, its own space-time points, and its own spatio-temporal relations. This make the argument against the $M$-property as a demarcation principle fall short. In contrast, our everyday claims concerning what could happen at a given location at a given time in an alternative course of events suggests that alternative scenarios share the same space-time. That idea of a shared space-time also underlies the mathematical description of indeterminism in classical physics. On this view, then, spatio-temporal relations relate incompatible events. Of course, we do not want to claim that any two incompatible events, which are spatio-temporally related in the wide sense, belong to our world. But some such events, like tomorrow's possible outcomes *tails up* and *heads up* of a particular coin toss do seem to belong to our world. (At least this is the intuition underlying BST.) The $M$-property, with the particular shape of the letter "$M$", should thus relate objects like the two incompatible results of a certain toss: these objects are incompatible, and yet they belong to our single, modally thick world.

## B.2 Answers to selected exercises from Chapter 2

**Exercise 2.2.** Prove the M property (Fact 2.4):
*For every pair $e_1, e_5$ of point events in $\mathscr{W}$, there are $e_2, e_3, e_4$ in $\mathscr{W}$ such that $e_1 \leqslant e_2$, $e_5 \leqslant e_4$ and $e_3 \leqslant e_2$, $e_3 \leqslant e_4$.*

*Proof.* Let $e_1 \in h_1$ and $e_5 \in h_5$. By Historical Connection (Postulate 2.2), there is $e_3 \in h_1 \cap h_5$. Since histories are directed, there is $e_2$ such that $e_1 \leqslant e_2$ and $e_3 \leqslant e_2$ (via directedness of $h_1$) and $e_5 \leqslant e_4$ and $e_3 \leqslant e_4$ (via directedness of $h_5$). □

**Exercise 2.3.** Let $\langle W, < \rangle$ be a partially ordered set satisfying Postulates 2.1 and 2.2. Prove that if every history of $W$ is downward directed, then so is $W$ as a whole. (Note that the assumption is true, for example, if each history is isomorphic to Minkowski space-time.)

*Proof.* Pick any $e_1, e_5 \in W$. If these events share a history, we are done. If not, let us invoke the $M$ property. There are thus $e_2, e_3, e_4$ such that $e_1 \leqslant e_2, e_5 \leqslant e_4$ and $e_3 \leqslant e_2, e_3 \leqslant e_4$. Let $e_2 \in h_2$ and $e_4 \in h_4$, so $e_3 \in h_2 \cap h_4$. By the assumption, histories are downward directed, so there are $e_6, e_7 \in h_2 \cap h_4$ that are lower bounds of, respectively, $e_1, e_3$ and $e_3, e_5$. Since $e_6, e_7$ share a history, they have a lower bound $e_9$. By transitivity of $\leqslant$, $e_9 \leqslant e_1$ and $e_9 \leqslant e_5$. □

## B.3 Answers to selected exercises from Chapter 3

**Exercise 3.2.** Prove the following extension of Fact 3.10: For a $\text{BST}_{92}$ structure $\langle W, < \rangle$ that has neither maximal nor minimal elements, its full transition structure $\langle W', <' \rangle =_{\text{df}} \langle \text{TR}_{\text{full}}(W), \prec \rangle$ has no maxima nor minima either.

*Proof.* For no maxima, let $\tau = e \rightarrowtail H \in W'$, and let $h \in H \subseteq H_e$. As $W$ contains no maxima, $h$ contains no maxima either (Fact 2.1(9)), so there is $e_1 \in h$ for which $e < e_1$. Accordingly we have $\tau' =_{\text{df}} e_1 \rightarrowtail \Pi_{e_1}\langle h \rangle \in W'$. It is easy to check that $\tau \prec \tau'$, which establishes that $\tau$ is not maximal in $W'$.

For no minima, similarly, let $\tau = e \rightarrowtail H \in W'$. As $W$ contains no minima, there is $e_1 \in W$ for which $e_1 < e$. Let $h \in H_e$. By downward closure, $e_1 \in h$, i.e., $h \in H_{e_1}$. So there is $\tau' =_{\text{df}} e_1 \rightarrowtail \Pi_{e_1}\langle h \rangle \in W'$, and $\tau' \prec \tau$. Thus, $\tau$ is not minimal in $W'$. □

**Exercise 3.7.** Let $\langle W, < \rangle$ satisfy Postulates 2.1–2.5. Let $l$ be an upper-bounded chain, and let $e =_{\text{df}} \sup_{h'}(l)$. Then for every history $h$ of $W$ containing the chain $l$, if $e$ lies in $h$, then $e = \sup_h(l)$.

*Proof.* Suppose that $e \in h$. Since $e = \sup_{h'}(l)$, by definition $e$ upper-bounds $l$. Now, suppose toward a contradiction that $e$ is not the least upper bound of $l$ in $h$, that is, suppose that there is some $e'$ in $h$ such that $l \leqslant e' < e$. By Fact 2.1 (5) histories are downwards closed, which means that the element $e'$ also lies in $h'$, contradicting $e = \sup_{h'}(l)$. Therefore there is no such $e'$ in $h$, and consequently, $e = \sup_h(l)$. □

## B.4 Answers to selected exercises from Chapter 4

**Exercise 4.4.** Prove that the diamond topology of Def. 4.13 and the history-relative diamond topologies of Def. 4.14 are indeed topologies for both $\mathrm{BST}_{92}$ and $\mathrm{BST}_{\mathrm{NF}}$; that is, prove that both (1) the base set ($W$ or $h$, respectively) and (2) the empty set are open, (3) arbitrary unions of open sets are open, and (4) finite intersections of open sets are open.

*Proof.* In both definitions, (1) is explicitly required to hold, and the form of the definitions, which is universal, guarantees that (2) and (3) hold as well (note that the condition is vacuous for the empty set). The only condition that needs a proof is the finite intersection property, (4). It suffices to prove that the intersection of any two open sets is open, as the finite case then follows by simple induction.

The proof of (4) is the same for $\mathrm{BST}_{92}$ and for $\mathrm{BST}_{\mathrm{NF}}$, as no instance of a prior choice principle is needed. We give the proof for $\mathscr{T}$; the proof for $\mathscr{T}_h$ is exactly analogous, replacing the set of chains $MC(e)$ with $MC_h(e)$. Let thus $Z_1, Z_2 \in \mathscr{T}$, let $Z =_{\mathrm{df}} Z_1 \cap Z_2$, and take some $e \in Z$ and $t \in MC(e)$. To show that $Z$ is open, we have to find $e_1, e_2 \in t$ with $e_1 < e < e_2$ and such that the diamond $D_{e_1,e_2} \subseteq Z$. Now as $Z_i$ is open ($i = 1.2$), there are $e_1^i, e_2^i$ with $e_1^i < e < e_2^i$ and such that $D_{e_1^i,e_2^i} \subseteq Z_i$. Let $e_1 = \max(e_1^1, e_1^2)$ and $e_2 = \min(e_1^1, e_1^2)$ (note that these elements are comparable as they belong to the same chain). Then $D_{e_1,e_2} \subseteq Z_1$ and $D_{e_1,e_2} \subseteq Z_2$, which implies that $D_{e_1,e_2} \subseteq Z$. This shows that $Z$ is indeed open. □

## B.5 Answers to selected exercises from Chapter 5

**Exercise 5.1.** Let $O$ be an outcome chain. Prove that if for all $e \in cll(O)$, we have $e < O$, then $H_{\langle O\rangle} = \bigcap_{e \in cll(O)} \Pi_e\langle O\rangle$.

*Proof.* For any $e \in cll(O)$, $e < O$ implies $H_{\langle O\rangle} \subseteq \Pi_e\langle O\rangle$ (see the proof of Fact 5.4), so the inclusion "$\subseteq$" is straightforward. For the opposite inclusion, we argue indirectly; that is, we assume for reductio that there is $h \in \mathrm{Hist}$ such that $h \notin H_{\langle O\rangle}$ but that for every $e \in cll(O)$, $h \in \Pi_e\langle O\rangle$. Take some $h_O \supseteq O$. As $h \notin H_{\langle O\rangle}$, we have $O \subseteq h_O \setminus h$, so by $\mathrm{PCP}_{92}$ there is $c \in W$ such that $c < O$ and $h \perp_c h_O$. By Fact 3.8, $h \perp_c H_{\langle O\rangle}$. Thus, $c \in cll(O)$, so by our assumption: $h \in \Pi_c\langle O\rangle$. Since $H_{\langle O\rangle} \subseteq \Pi_c\langle O\rangle$, we get $h \equiv_c H_{\langle O\rangle}$, which contradicts $h \perp_c H_{\langle O\rangle}$. □

**Exercise 5.3.** Prove a version of Facts 5.4 and 4.7(2) for $\hat{O}$ a scattered outcome. That is, prove the following facts:
*Let $\hat{O}$ be a scattered outcome. (1) If there is no-MFB, then for all $e \in cll(\hat{O})$, we have $e < \hat{O}$. (2) If $e < \hat{O}$, then there is a unique basic outcome of $e$ that is consistent with $H_{\langle\hat{O}\rangle}$, which we denote $\Pi_e\langle\hat{O}\rangle$.*

*Proof.* (1) By no-MFB we know from Fact 5.4 that for each $O \in \hat{O}$, any $e \in cll(O)$ is in the past of $O$. By the same Fact and by Exercise 5.1, for any $O \in \hat{O}$ we have $H_{\langle O \rangle} = \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle$. Thus, by the definition of $H_{\langle \hat{O} \rangle}$,

$$H_{\langle \hat{O} \rangle} = \left( \bigcap_{O \in \hat{O}} \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle \right). \tag{B.1}$$

Let us now suppose for reductio that there is $h \in \text{Hist}$ and $c \in W$ such that $h \perp_c H_{\langle \hat{O} \rangle}$ (i.e., $c \in cll(\hat{O})$) but $c \not< \hat{O}$, which means that for every $O \in \hat{O}$: $c \not< O$. Accordingly, $c \notin \bigcup_{O \in \hat{O}} cll(O)$ but $c \in cll(\hat{O})$. Consider now the union of outcomes of $c$ that are consistent with $H_{\langle \hat{O} \rangle}$, which is $\tilde{H} =_{\text{df}} \bigcup \{H \in \Pi_c \mid H \cap H_{\langle \hat{O} \rangle} \neq \emptyset\}$. Observe next that since $h \perp_c H_{\langle \hat{O} \rangle}$, we have $H_{\langle \hat{O} \rangle} \subseteq H_c$, and hence $H_{\langle \hat{O} \rangle} \subseteq \tilde{H}$. Note also that for every $O \in \hat{O}$ and for every $e \in cll(O)$, $e < O$, so that $H_{\langle O \rangle} \subseteq \Pi_e \langle O \rangle$. Accordingly, for every $O \in \hat{O}$ and for every $e \in cll(O)$, we have $H_{\langle \hat{O} \rangle} \subseteq H_{\langle O \rangle} \subseteq \Pi_e \langle O \rangle$. Now consider the following intersection of sets of histories, $H'$:

$$H' =_{\text{df}} \left( \bigcap_{O \in \hat{O}} \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle \right) \cap \tilde{H}. \tag{B.2}$$

By $H_{\langle \hat{O} \rangle} \subseteq \tilde{H}$ and by Eq. (B.1) we have $H_{\langle \hat{O} \rangle} = H'$, i.e.,

$$\left( \bigcap_{O \in \hat{O}} \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle \right) \cap \tilde{H} = H_{\langle \hat{O} \rangle}. \tag{B.3}$$

Since $\Pi_c \langle h \rangle \subseteq H_c \setminus \tilde{H}$, Eqs. (B.3) and (B.1) imply

$$\left( \bigcap_{O \in \hat{O}} \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle \right) \cap \Pi_c \langle h \rangle = \emptyset. \tag{B.4}$$

We now claim that Eq. (B.4) implies that our structure contains an instance of modal funny business, which contradicts our premise of no-MFB.

Let $\tau_c =_{\text{df}} (c \rightarrowtail \Pi_c \langle h \rangle)$, and consider the following set of transitions, which we claim is combinatorially consistent (Def. 5.5):

$$T =_{\text{df}} \{e \rightarrowtail \Pi_e \langle O \rangle \mid O \in \hat{O}, e \in cll(O)\} \cup \{\tau_c\}.$$

The subset to the left is combinatorially consistent since it is consistent by Eq. (B.1). So to check combinatorial consistency, we only need to consider pairs of $\tau_c$ and some $\tau_e = (e \rightarrowtail \Pi_e \langle O \rangle)$, for some $O \in \hat{O}$ and some $e \in cll(O)$. Note that for any $h' \in H_{\langle \hat{O} \rangle}$, $\{e, c\} \subseteq h'$. It follows that if $e < c$, then $\Pi_c \langle h \rangle \subseteq \Pi_e \langle O \rangle$, and if $e, c$ are incomparable, then they are *SLR*. Finally, it cannot happen that $c \leqslant e$, since this implies $c < \hat{O}$, contrary to our assumption that $c \not< \hat{O}$. Thus, $T$ is combinatorially consistent, but inconsistent (by Eq. (B.4)). This shows that there is CFB in the structure, contrary to the premise of no-MFB. We thus have established that any element of $cll(\hat{O})$ is in the past of $\hat{O}$.

(2) Let $e \in cll(\hat{O})$ and $e < \hat{O}$, i.e., there is some $O \in \hat{O}$ for which $e < O$. Then by Fact 4.7(2), $\Pi_e\langle O\rangle$ is uniquely defined, and we can set $\Pi_e\langle\hat{O}\rangle =_{\mathrm{df}} \Pi_e\langle O\rangle$, as $H_{\langle\hat{O}\rangle} \subseteq H_{\langle O\rangle} \subseteq \Pi_e\langle O\rangle$. □

## B.6 Answers to selected exercises from Chapter 6

**Exercise 6.1.** Prove clause (2) of Fact 6.1.

That is, given a transition $I \rightarrowtail \hat{O}$ to a scattered outcome $\hat{O}$, we have to prove $cll(I \rightarrowtail \hat{O}) = \bigcup_{O\in\hat{O}} cll(I \rightarrowtail O)$.

*Proof.* "$\supseteq$": Let $O \in \hat{O}$, and let $e \in cll(I \rightarrowtail O)$. Then for $h \in H_{[I]}$, $h \perp_e H_{\langle O\rangle}$. Since $H_{\langle\hat{O}\rangle} \subseteq H_{\langle O\rangle}$, we have $h \perp_e H_{\langle\hat{O}\rangle}$, and hence $e \in cll(I \rightarrowtail \hat{O})$.

"$\subseteq$": As there is no MFB in $\mathscr{W}$, for every $e \in cll(I \rightarrowtail \hat{O})$ we have $e < \hat{O}$ (see Exercise 5.3). Thus, $e \in cll(I \rightarrowtail \hat{O})$ implies that there is $O \in \hat{O}$ for which $e < O$. Let $h' \in H_{\langle\hat{O}\rangle} \subseteq H_{\langle O\rangle}$, and let $h'' \in H_{\langle O\rangle}$. We have $h' \equiv_e h''$. As $h \perp_e h'$ for some $h \in H_{[I]}$, it follows that $h \perp_e H_{\langle O\rangle}$, and hence $e \in cll(I \rightarrowtail O)$. □

**Exercise 6.2.** Prove clauses (4) and (5) of Fact 6.4.

That is, making no assumptions about the presence or absence of MFB and given $e \in cll(I \rightarrowtail \mathscr{O}^*)$, (4) for $\mathscr{O}^* = \hat{O}$ a scattered outcome, we have to show that there is some initial segment $O'$ of some $O \in \hat{O}$ such that for every $e' \in O'$, we have $e \leqslant e'$ or $e\,SLR\,e'$, and (5) for $\mathscr{O}^* = \breve{\mathbf{O}}$ a disjunctive outcome, we have to show that there is some $\hat{O} \in \breve{\mathbf{O}}$ and some initial segment $O'$ of some $O \in \hat{O}$ such that for every $e' \in O'$, we have $e \leqslant e'$ or $e\,SLR\,e'$,

*Proof.* (4) Let $e \in cll(I \rightarrowtail \hat{O})$. If $e \in cll(I \rightarrowtail O)$ for some $O \in \hat{O}$, then by clause (3) of Fact 6.4, for every $e' \in O$ either $e < e'$ or $e\,SLR\,e'$, so we are done. The other case is that $e \in cll(I \rightarrowtail \hat{O})$ but $e \notin cll(I \rightarrowtail O)$ for all $O \in \hat{O}$. Note that $cll(I \rightarrowtail \hat{O})$ is consistent, and pick some $O \in \hat{O}$ such that for some $h \in H_{[I]}: h \cap O = \emptyset$; thus by $\mathrm{PCP}_{92}$ there is $c < O$ such that $h \perp_c H_{\langle O\rangle}$, and hence $c \in cll(I \rightarrowtail O) \subseteq cll(I \rightarrowtail \hat{O})$. Since elements of $cll(I \rightarrowtail \hat{O})$ are SLR, we have that (†) $e\,SLR\,c$.

As $e \in cll(I \rightarrowtail \hat{O})$, there is some $h$ in $H_{[I]}$ for which $h \perp_e H_{\langle\hat{O}\rangle}$. Pick some $h_O \in H_{\langle\hat{O}\rangle}$. Thus, in particular, $h_O \in H_{\langle O\rangle}$. We have $e \in h_O$ by $h \perp_e H_{\langle\hat{O}\rangle}$, and as $h_O \in H_{\langle O\rangle}$, there is an initial segment $O' \subseteq O$ such that $O \subseteq h_O$. As $c < O$, we have $c < O$, and $c \in h_O$ holds by downward closure of histories. The claim then follows: for $e \in O$, we have $e, e \in h_O$, so $e$ and $e'$ are either comparable or $SLR$. It cannot be that $e \leqslant e$, as from $c < e'$ we could conclude $c < e$, contradicting (†), so either $e < e$ or $e\,SLR\,e'$.

(5) follows immediately from the above by noting that if $e \in cll(I \rightarrowtail \breve{\mathbf{O}})$, then $e \in cllr(I \rightarrowtail \hat{O}) \subseteq cll(I \rightarrowtail \hat{O})$ for some $\hat{O} \in \breve{\mathbf{O}}$. □

**Exercise 6.6.** Prove Theorem 6.3.

**Theorem 6.3.** (nns for transitions to outcome chains or scattered outcomes in $\mathrm{BST}_{92}$ with MFB) *Let $\mathscr{O}^*$ be an outcome chain or a scattered outcome. Then the* causae causantes *of $I \rightarrowtail \mathscr{O}^*$ satisfy the following inus-related conditions:*

1. *joint sufficiency – nns:* $\bigcap_{e\in cll(I\rightarrowtail\mathscr{O}^*)} H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle} \subseteq H_{I\rightarrowtail\mathscr{O}^*}$;
2. *joint necessity – nns:* $H_{\langle\mathscr{O}^*\rangle} = H_{[I]}\cap H_{I\rightarrowtail\mathscr{O}^*} \subseteq \bigcap_{e\in cll(I\rightarrowtail\mathscr{O}^*)} H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle}$;
3. *non-redundancy – nns: for every* $(e_0\rightarrowtail\breve{\mathbf{H}})\in CC(I\rightarrowtail\mathscr{O}^*)$ *and every* $\breve{\mathbf{H}}'$ *such that* $\breve{\mathbf{H}}\cap\breve{\mathbf{H}}'=\emptyset$, *where* $\breve{\mathbf{H}},\breve{\mathbf{H}}'\subseteq\Pi_{e_0}$

$$\text{either}\quad \bigcup\breve{\mathbf{H}}'\cap\bigcap_{e\in cll(I\rightarrowtail\mathscr{O}^*)\setminus\{e_0\}}(H_e\cap H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle})=\emptyset,\quad\text{or}\tag{6.5}$$

$$\bigcup\breve{\mathbf{H}}'\cap\bigcap_{e\in cll(I\rightarrowtail\mathscr{O}^*)\setminus\{e_0\}}(H_e\cap H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle})\not\subseteq H_{[I]}\cap H_{I\rightarrowtail\mathscr{O}^*}.\tag{6.6}$$

*Proof.* We give a proof for transitions to scattered outcomes, which can easily be simplified to cover transitions to outcome chains as well.

(1) If $I\rightarrowtail\mathscr{O}^*$ is deterministic, then the theorem holds as then $H_{I\rightarrowtail\mathscr{O}^*}=$ Hist. We thus assume that the transition is indeterministic (i.e., $H_{[I]}\setminus H_{\langle\mathscr{O}^*\rangle}\neq\emptyset$) and argue for the contraposition, so we take $h\in H_{[I]}\setminus H_{\langle\mathscr{O}^*\rangle}$, and hence we get that for some $c$: $h\perp_c H_{\langle\mathscr{O}^*\rangle}$. Accordingly, $c\in cll(I\rightarrowtail\mathscr{O}^*)$. Further, for every $H\in\breve{\mathbf{H}}_c\langle H_{\langle\mathscr{O}^*\rangle}\rangle$: $h\perp_c H$. Thus, for every $H$ of that sort, $h\notin H$, and hence $h\notin\bigcup\breve{\mathbf{H}}_c\langle H_{\langle\mathscr{O}^*\rangle}\rangle$. Since $h\in H_c$, it follows that $h\notin H_{c\rightarrowtail\breve{\mathbf{H}}_c\langle H_{\langle\mathscr{O}^*\rangle}\rangle}$.

(2) Pick an arbitrary $h\in H_{\langle\mathscr{O}^*\rangle}$ and an arbitrary $e\in cll(I\rightarrowtail\mathscr{O}^*)$. Clearly, as $h\in H_e$, $h$ belongs to some $H\in\Pi_e$ such that $H\cap H_{\langle\mathscr{O}^*\rangle}\neq\emptyset$, Thus $h\in\bigcup\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle$, and hence $h\in H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle}$.

(3) Since $(e_0\rightarrowtail\breve{\mathbf{H}})\in CC(I\rightarrowtail\mathscr{O}^*)$, it must be that $\breve{\mathbf{H}}=\breve{\mathbf{H}}_{e_0}\langle H_{\langle\mathscr{O}^*\rangle}\rangle$. Pick then an arbitrary $\breve{\mathbf{H}}'\subseteq\Pi_{e_0}$ such that $\breve{\mathbf{H}}'\cap\breve{\mathbf{H}}_{e_0}\langle H_{\langle\mathscr{O}^*\rangle}\rangle=\emptyset$. Hence $(\bigcup\breve{\mathbf{H}}')\cap(\bigcup\breve{\mathbf{H}}_{e_0}\langle H_{\langle\mathscr{O}^*\rangle}\rangle)=\emptyset$. By Fact 6.3 we have $H_{\langle\mathscr{O}^*\rangle}\subseteq\bigcup\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle$, which implies $(*)$ $\bigcup\breve{\mathbf{H}}'\cap H_{\langle\mathscr{O}^*\rangle}=\emptyset$. Let us abbreviate $H^-=\bigcap_{e\in cll(I\rightarrowtail\mathscr{O}^*)\setminus\{e_0\}}H_e\cap H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\mathscr{O}^*\rangle}\rangle}$ and consider two cases: (i) $\bigcup\breve{\mathbf{H}}'\cap H^-=\emptyset$ and (ii) $\bigcup\breve{\mathbf{H}}'\cap H^-\neq\emptyset$. If (i), we are done. If (ii), by $(*)$ we have $H^-\cap(\bigcup\breve{\mathbf{H}}')\cap H_{\langle\mathscr{O}^*\rangle}=\emptyset$. Since $H_{\langle\mathscr{O}^*\rangle}\neq\emptyset$, it follows that $H^-\cap(\bigcup\breve{\mathbf{H}}')\not\subseteq H_{\langle\mathscr{O}^*\rangle}$, which is Eq. 6.6. □

**Exercise 6.7.** Prove Theorem 6.4.

**Theorem 6.4.** (nus for transitions to disjunctive outcomes in $BST_{92}$ with MFB) *Let* $\breve{\mathbf{O}}=\{\hat{O}_\gamma\mid\gamma\in\Gamma\}$ *be a disjunctive outcome consisting of more than one scattered outcome. The set of* causae causantes *of* $I\rightarrowtail\breve{\mathbf{O}}$, *i.e.,* $\{CCr(I\rightarrowtail\hat{O}_\gamma)\}_{\gamma\in\Gamma}$ *as well as each* $CC(I\rightarrowtail\hat{O}_\gamma)$, *satisfy the following inus-related conditions:*

1. *each* $CCr(I\rightarrowtail\hat{O}_\gamma)$ *is sufficient – nus: for every* $\gamma\in\Gamma$:
   $\bigcap_{e\in cllr(I\rightarrowtail\hat{O}_\gamma)}H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\hat{O}_\gamma\rangle}\rangle}\subseteq H_{I\rightarrowtail\breve{\mathbf{O}}}$;
2. *each* $CC(I\rightarrowtail\hat{O}_\gamma)$ *is unnecessary – nus: for every* $\gamma\in\Gamma$:
   $H_{\langle\breve{\mathbf{O}}\rangle}=H_{[I]}\cap H_{I\rightarrowtail\breve{\mathbf{O}}}\not\subseteq\bigcap_{e\in cll(I\rightarrowtail\hat{O}_\gamma)}H_{e\rightarrowtail\breve{\mathbf{H}}_e\langle H_{\langle\hat{O}_\gamma\rangle}\rangle}$.
3. *for each* $\gamma\in\Gamma$, *each* $\tau_0=(e_0\rightarrowtail\breve{\mathbf{H}})\in CCr(I\rightarrowtail\hat{O}_\gamma)$ *is non-redundant – nus. That is, for every* $\breve{\mathbf{H}}'\subseteq\Pi_{e_0}$ *such that* $\breve{\mathbf{H}}\cap\breve{\mathbf{H}}'=\emptyset$:

$$\textit{either} \quad \bigcup \breve{\mathbf{H}}' \cap \bigcap_{e \in cllr(I \rightarrowtail \hat{O}_\gamma) \setminus \{e_0\}} (H_e \cap H_{e \rightarrowtail \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle}) = \emptyset, \tag{6.7}$$

$$\textit{or} \quad \bigcup \breve{\mathbf{H}}' \cap \bigcap_{e \in cllr(I \rightarrowtail \hat{O}_\gamma) \setminus \{e_0\}} (H_e \cap H_{e \rightarrowtail \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle}) \not\subseteq H_{[I]} \cap H_{I \rightarrowtail \hat{O}_\gamma}. \tag{6.8}$$

*Proof.* (1) This is almost exactly the same proof as for Theorem 6.2(1). Just like in this proof, from $h \notin H_{\hat{O}_\gamma}$ we arrive at $h \perp_e \Pi_e \langle \hat{O}_\gamma \rangle$ and get that $e \notin DET_{I \rightarrowtail \breve{\mathbf{O}}}$ since $h \in (H_e \cap H_{[I]}) \setminus H_{\langle \breve{\mathbf{O}} \rangle}$, which entails $e \in cllr(I \rightarrowtail \hat{O}_\gamma)$. As we allow for MFB, $h \notin H_{\langle \hat{O}_\gamma \rangle}$ implies $h \notin \{H \in \Pi_e \mid H \cap H_{\langle \hat{O}_\gamma \rangle}\} = \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle$. Accordingly, $h \notin H_{e \rightarrowtail \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle}$. (2) Since $\breve{\mathbf{O}}$ has at least two scattered outcomes, there is $\hat{O}_\gamma \in \breve{\mathbf{O}}$ and $h \in H_{[I]}$ such that $h \notin H_{\langle \hat{O}_\gamma \rangle}$. By Theorem 6.3(1) $h \notin \bigcap_{e \in cll(I \rightarrowtail \hat{O}_\gamma)} H_{e \rightarrowtail \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle}$.

(3) Pick an arbitrary $\gamma \in \Gamma$ and an arbitrary $(e_0 \rightarrowtail \breve{\mathbf{H}}) \in CCr(I \rightarrowtail \hat{O}_\gamma)$, so $cllr(I \rightarrowtail \hat{O}_\gamma) \neq \emptyset$. It must be that $\breve{\mathbf{H}} = \breve{\mathbf{H}}_{e_0} \langle H_{\langle \hat{O}_\gamma \rangle} \rangle$. Pick then a $\breve{\mathbf{H}}' \subseteq \Pi_{e_0}$ such that $\breve{\mathbf{H}}' \cap \breve{\mathbf{H}}_{e_0} \langle H_{\langle \hat{O}_\gamma \rangle} \rangle = \emptyset$. Hence $(\bigcup \breve{\mathbf{H}}') \cap (\bigcup \breve{\mathbf{H}}_{e_0} \langle H_{\hat{O}_\gamma} \rangle) = \emptyset$. By Fact 6.3 we have $H_{\langle \hat{O}_\gamma \rangle} \subseteq \bigcup \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle$, which implies $(*)$ $\bigcup \breve{\mathbf{H}}' \cap H_{\langle \hat{O}_\gamma \rangle} = \emptyset$. Let us abbreviate $H^- =_{\mathrm{df}} \bigcap_{e \in cllr(I \rightarrowtail \mathscr{O}^*) \setminus \{e_0\}} H_e \cap H_{e \rightarrowtail \breve{\mathbf{H}}_e \langle H_{\langle \hat{O}_\gamma \rangle} \rangle}$ and consider two cases: (i) $\bigcup \breve{\mathbf{H}}' \cap H^- = \emptyset$ and (ii) $\bigcup \breve{\mathbf{H}}' \cap H^- \neq \emptyset$. If (i), we are done. If (ii), by $(*)$ we have $H^- \cap (\bigcup \breve{\mathbf{H}}') \cap H_{\langle \hat{O}_\gamma \rangle} = \emptyset$. Since $H_{\langle \hat{O}_\gamma \rangle} \neq \emptyset$, it follows that $H^- \cap (\bigcup \breve{\mathbf{H}}') \not\subseteq H_{\langle \hat{O}_\gamma \rangle}$, which is Eq. 6.8. □

**Exercise 6.9.** Prove Theorem 6.5

**Theorem 6.5.** *(nns for transitions to outcome chains or scattered outcomes in $BST_{NF}$ with no MFB) Let $\mathscr{O}^*$ be an outcome chain or a scattered outcome. The* causae causantes *of $I \rightarrowtail \mathscr{O}^*$ satisfy the following inus-related conditions:*

1. *joint sufficiency – nn$\underline{s}$: $\bigcap_{\ddot{e} \in cll(I \rightarrowtail \mathscr{O}^*)} H_{\ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \mathscr{O}^* \rangle} \subseteq H_{I \rightarrowtail \mathscr{O}^*}$;*
2. *joint necessity – n$\underline{n}$s: $H_{\langle \mathscr{O}^* \rangle} = H_{[I]} \cap H_{I \rightarrowtail \mathscr{O}^*} \subseteq \bigcap_{\ddot{e} \in cll(I \rightarrowtail \mathscr{O}^*)} H_{\ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \mathscr{O}^* \rangle}$;*
3. *non-redundancy – $\underline{n}$ns: for every $(\ddot{e}_0 \rightarrowtail H) \in CC(I \rightarrowtail \mathscr{O}^*)$ and every $H' \in \Pi_{\ddot{e}_0}$ such that $H' \cap H = \emptyset$:*

$$\textit{either } H' \cap \bigcap_{\ddot{e} \in cll(I \rightarrowtail \mathscr{O}^*) \setminus \{\ddot{e}_0\}} \Pi_{\ddot{e}} \langle \mathscr{O}^* \rangle = \emptyset, \tag{6.9}$$

$$\textit{or} \quad H' \cap \bigcap_{\ddot{e} \in cll(I \rightarrowtail \mathscr{O}^*) \setminus \{\ddot{e}_0\}} \Pi_{\ddot{e}} \langle \mathscr{O}^* \rangle \not\subseteq H_{[I]} \cap H_{I \rightarrowtail \mathscr{O}^*}. \tag{6.10}$$

*Proof.* (1) If $H_{[I]} = H_{\langle \hat{O} \rangle}$, then $H_{I \rightarrowtail \hat{O}} = \text{Hist}$, and we are done. Let us thus suppose that there is $h \in \text{Hist}$ such that $h \in H_{[I]}$ but $h \notin H_{\langle \hat{O} \rangle}$. Thus, there is $O \in \hat{O}$ s.t $h \cap O = \emptyset$. There is also $h' \in H_{\langle \hat{O} \rangle}$, which implies $h' \in H_{[I]}$ and $h' \in H_{\langle O \rangle}$. Accordingly, there is $O' \subseteq O$ such that $O' \subseteq h' \setminus h$. By $\text{PCP}_{\text{NF}}$ there is $\ddot{c}$, with $c \in \ddot{c}$ and $c \leqslant O'$ such that $h \perp_{\ddot{c}} H_{\langle O' \rangle}$. Since $H_{\langle \hat{O} \rangle} \subseteq H_{\langle O \rangle} \subseteq H_{\langle O' \rangle}$, we get $h \perp_{\ddot{c}} H_{\langle \hat{O} \rangle}$; thus $\ddot{c} \in cll(I \rightarrowtail \hat{O})$. On the other hand, $h \in H_{\ddot{c}}$, and $h \perp_{\ddot{c}} H_{\langle \hat{O} \rangle}$ implies $h \notin \Pi_{\ddot{c}} \langle \hat{O} \rangle$. For if $h \in \Pi_{\ddot{c}} \langle \hat{O} \rangle$, then (since $c \leqslant \hat{O}$) $h \equiv_{\ddot{c}} h_1$ for every $h_1 \in H_{\langle \hat{O} \rangle}$, contradicting $h \perp_{\ddot{c}} H_{\langle \hat{O} \rangle}$. Hence $h \notin H_{\ddot{c} \rightarrowtail \Pi_{\ddot{c}} \langle \hat{O} \rangle}$. By simplifying this proof appropriately, one obtains the argument for transitions to outcome chains.

(2) Note that $H_{[I]} \cap H_{I \rightarrowtail \mathscr{O}^*} = H_{\langle \mathscr{O}^* \rangle}$ and for every $c \leqslant \mathscr{O}^*$: $H_{\langle \mathscr{O}^* \rangle} \subseteq H_c \subseteq \Pi_{\ddot{c}}\langle \mathscr{O}^* \rangle$. Thus, $H_{[I]} \cap H_{I \rightarrowtail \mathscr{O}^*} \subseteq H_{\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle \mathscr{O}^* \rangle}$.

(3) Pick an arbitrary $H'$ such that $H' \cap H = \emptyset$, where $H, H' \in \Pi_{\ddot{e}_0}$ and $\ddot{e}_0 \rightarrowtail H \in CC(I \rightarrowtail \hat{O})$. Since $H_{\langle \hat{O} \rangle} \subseteq H$, $(*)$ $H' \cap H_{\langle \hat{O} \rangle} = \emptyset$. Let us next abbreviate: $H^- = \bigcap_{\ddot{e} \in cll(I \rightarrowtail \hat{O}) \setminus \{\ddot{e}_0\}} \Pi_{\ddot{e}}\langle \hat{O} \rangle$ and consider then two cases: (i) $H' \cap H^- = \emptyset$ and (ii) $H' \cap H^- \neq \emptyset$. If (i), since it is identical to Eq. 6.9, we are done. In (ii), by $(*)$ we have $H' \cap H^- \cap H_{\langle \hat{O} \rangle} = \emptyset$. Since $H_{\langle \hat{O} \rangle} \neq \emptyset$, it follows that $H' \cap H^- \not\subseteq H_{\langle \hat{O} \rangle}$, which is Eq. 6.10. □

## B.7 Answers to selected exercises from Chapter 7

**Exercise 7.1.** Prove Lemma 7.2.

**Lemma 7.2.** *Let the conditions of Def. 7.5 hold for a transition $I \rightarrowtail \check{\mathbf{O}}$ to a disjunctive outcome $\check{\mathbf{O}}$, and consider $CPS(I \rightarrowtail \check{\mathbf{O}}) = \langle S, \mathscr{A}, p \rangle$. That triple is in fact a probability space satisfying Def. 7.1. That is, $CPS(I \rightarrowtail \check{\mathbf{O}})$ is well defined and $p$ is a normalized measure on $\mathscr{A}$. Furthermore, we have that*

$$p(CC(I \rightarrowtail \hat{O}_\gamma)) = \mu(\{T \in S \mid CC(I \rightarrowtail \hat{O}_\gamma) \subseteq T\}) = \sum_{T \in S,\, CC(I \rightarrowtail \hat{O}_\gamma) \subseteq T} \mu(T);$$

$$p(CC(I \rightarrowtail \check{\mathbf{O}})) = \sum_{\gamma \in \Gamma} p(CC(I \rightarrowtail \hat{O}_\gamma)).$$

*Proof.* The observation underlying this proof is that $T \in S$ iff $T \in S_\gamma$ for some $\gamma$ such that $\hat{O}_\gamma \in \check{\mathbf{O}}$, and where $S_\gamma$ is the set of causal alternatives to $CC(I \rightarrowtail \hat{O}_\gamma)$.

We first prove that for every $T \in S$, $\mu(T)$ is defined. If $T \in S$ then $T \in S_\gamma$ for some $\gamma$ such that $\hat{O}_\gamma \in \check{\mathbf{O}}$, so by Def 7.5 and Postulate 7.2, $\mu(T)$ is defined. Thus $p(T)$ is defined via Def. 7.4, which induces measure $p$ on the whole $\mathscr{A}$.

We need to show that $p$ is a normalized probability measure. It suffices to show that the probabilities assigned to the different elements of $S$ sum to one, as then

$$p(\mathbf{1}_{\mathscr{A}}) = \sum_{T \in S} p(T) = 1.$$

Our proof uses the law of total probability in the form of Postulate 7.3. We thus have to show that the elements of $S$ partition the set of histories in which the initial $I$ occurs. First, we have to show that for two different $T_1, T_2 \in S$, $T_1 \neq T_2$, we have $H(T_1) \cap H(T_2) = \emptyset$. If $T_1, T_2 \in S_\gamma$, the claim follows immediately since $T_1, T_2$ are maximal consistent subsets. If there is no $\hat{O}_\gamma \in \check{\mathbf{O}}$ such that $T_1, T_2 \in\in \hat{O}_\gamma$, the claim follows from the fact that elements of $\check{\mathbf{O}}$ are inconsistent as $\check{\mathbf{O}}$ is a disjunctive outcome.

Second, we show $H_{[I]} \subseteq \bigcup_{T \in S} H(T)$. In the proof of Lemma 7.1, we already showed that $H_{[I]} \subseteq \bigcup_{T \in S_\gamma} H(T)$. Since $S_\gamma \subseteq S$, the claim follows. With the premises of the law of total causal probability (Postulate 7.3) satisfied, we therefore have

$$\sum_{T \in S} p(T) = \sum_{T \in S} \mu(T) = 1,$$

showing that the measure $p$ is indeed normalized. □

## B.8 Answers to selected exercises from Chapter 8

**Exercise 8.3.** Prove Fact 8.11.

*Proof.* Since each $T \in S$ is maximal consistent, for any two $T_1, T_2 \in S$, $T_1$-histories and $T_2$-histories have to split. Pick $T_1 \in S$. If for any other $T_2 \in S$, $T_1$-histories split with $T_2$-histories at members of $E$ only, then $\{T_1\} = \lambda \in \mathfrak{I}_c$, and hence $T_1 \in \bigcup \lambda$. Suppose there is $T_2 \in S$ such that $T_1$-histories and $T_2$-histories split at a member of $C$. If for any other $T_3 \in S$, $T_3$-histories split with $T_1$-histories at $E$, or $T_3$-histories split with $T_2$-histories at $E$, we put $\{T_1, T_2\} = \lambda$. If not, we consider a triple. Given the finiteness of $S$, we will find a maximal subset of $S$ with respect to the defining condition for contextual instruction sets. □

## B.9 Answers to selected exercises from Chapter 9

**Exercise 9.3.** Furnish the detail of the chain construction in the proof of Fact 9.2(2).

*Proof.* We define a function '*up*': for $a = \langle a^0, a^1, a^2, a^3 \rangle \in \mathbb{R}^4$, $b = \langle b^0, b^1, b^2, b^3 \rangle \in \mathbb{R}^4$, we let $up(a,b) =_{\text{df}} \langle a^0 + (\sum_1^3 (a^i - b^i)^2)^{1/2},\ a^1, a^2, a^3 \rangle \in \mathbb{R}^4$.

The chain $E$ is constructed in the following way:

*Step 1.* $z_0 = up(y, x_0)$, $z_1 = up(z_0, x_1)$, and generally, $z_{k+1} = up(z_k, x_{k+1})$. Note that $x_k = T \circ \Theta(\sigma_k)$.

*Step 2.* Suppose $\rho$ is a limit ordinal. Define $A_\rho := \{z_m \mid m < \rho\}$. $A_\rho$ is the part of our chain we have managed to construct so far. We need to distinguish two cases:

Case 1: $A_\rho$ is upper bounded with respect to $\leqslant_M$. Then it has to have 'vertical' upper bounds $t_0, t_1, \ldots$ with spatial coordinates $t_n^i = z_0^i$ $(i = 1,2,3)$. In this case, we use the function $T$ to choose one of the upper bounds of $A_\rho$:

$$t_\rho := T(\{t \in \mathbb{R}^4 \mid \forall m < \rho\ z_m \leqslant_M t \wedge t^i = z_0^i (i = 1,2,3)\}). \tag{B.5}$$

Then we put $z_\rho =_{\text{df}} up(t_\rho, x_\rho)$, arriving at the next element of our chain $E$.

Case 2: If $A_\rho$ is not upper bounded with respect to $\leqslant_M$, then no matter which point in $\mathbb{R}^4$ we choose, it is possible to find a point from $A_\rho$ above it (since $A_\rho$ is time-like). Therefore, the set

$$B_\rho = \{t \in A_\rho \mid x_\rho \leqslant_M t\} \tag{B.6}$$

is not empty. We put $z_\rho =_{\text{df}} T(B_\rho)$, arriving at the next element of our chain $E$. □

## B.10 Answers to selected exercises from Chapter 10

**Exercise 10.2.** Prove the strengthened version of identity (∗) from the proof of Fact 10.1, which restricts *cll* of *e* to those lying in the past of *e* (i.e., prove Fact 10.11):
For any $e \in W$,

$$H_e = \bigcap \{\Pi_{\ddot{c}} \langle H_e \rangle \mid \ddot{c} \in cll(e) \wedge \exists c \in \ddot{c}\ [c \leqslant e]\}. \tag{*}$$

*Proof.* For the "$\subseteq$" direction, since $c \leqslant e$, $H_e \subseteq H_c$, and $H_c$ is identical to $\Pi_{\ddot{c}}\langle H_e\rangle$, hence $H_e \subseteq \Pi_{\ddot{c}}\langle H_e\rangle$.

For the "$\supseteq$" direction, let us assume for reductio that there is (†) $h \in \bigcap\{\Pi_{\ddot{c}}\langle H_e\rangle \mid \ddot{c} \in cll(e) \wedge \exists c \in \ddot{c}\ c \leqslant e\}$, but $h \notin H_e$. Take some $h' \in H_e$. As $e \in h' \setminus h$, by $\text{PCP}_{\text{NF}}$ there is a choice set $\ddot{c}$ at which $h \perp_{\ddot{c}} h'$, and $c \in \ddot{c}$ such that $c \leqslant e$. From the last relations $h \perp_{\ddot{c}} H_e$ follows, so that $\ddot{c} \in cll(e)$. Since $h \perp_{\ddot{c}} H_e$, we get $h \notin \Pi_{\ddot{c}}\langle H_e\rangle$, and hence $h \notin \bigcap_{\ddot{c}\in cll(e)} \Pi_{\ddot{c}}\langle H_e\rangle$, which contradicts (†). □