11
Absolute Generality and Singularization

11.1 Absolute generality

Is it possible to assert something of absolutely everything there is? It certainly *seems* so; consider for instance the following assertions:

(11.1) Everything is physical.

(11.2) The empty set has no elements.

The truth of these assertions, it seems, rules out the existence of absolutely any ghost or element of the empty set. Any ghost or element of the empty set, no matter how remote or unfamiliar, is incompatible with what has been asserted. Let *absolute generality* be the view that it is possible to quantify over absolutely everything there is.

Plausible though it appears, absolute generality faces some challenges. We begin by laying out what we take to be the most interesting and powerful one. We do this in some detail, as it will be important to understand exactly which options we have for responding. We then argue, following Williamson (2003), that the rejection of absolute generality faces serious expressibility problems. Next, we examine Williamson's defense of absolute generality, which gives up the thesis that there are universal devices of singularization. We show how this proposal leads to an ascent to languages of ever higher orders and argue that the resulting outlook suffers from expressibility problems that are very similar to those that Williamson sought to avoid.[1] Motivated by this, we explore an alternative approach to the challenge, which allows a form of absolute generality but denies that the associated domain is extensionally definite (that is, properly circumscribed), and on this basis denies that the domain is an all-encompassing plurality of objects.

Absolute generality has emerged as one of the central themes in the book, figuring as an essential premise in several arguments in the preceding

[1] Here we draw on arguments from Linnebo 2006 and Linnebo and Rayo 2012.

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chapters. For instance, absolute generality was crucial to the more promising attempts to refute regimentation singularism. Moreover, it was invoked in the comparisons between alternative ways to talk about the many; for example, it emerged as an obstacle to the elimination of pluralities in favor of sets and to the mereological analysis of plurals. Furthermore, absolute generality was proposed as a constraint in the choice of a model theory for plural logic, a constraint that would rule out any form of semantic singularism cashed out in first-order terms. In order to reach a verdict on all these arguments, we need to resolve the question of absolute generality.

11.2 A challenge to absolute generality

Let us now formulate a challenge to absolute generality. The challenge is based on the plural version of Cantor's theorem that we presented in Section 3.5.

> Plural Cantor
>
> For any plurality *xx* with two or more members, the subpluralities of *xx* are strictly more numerous than the members of *xx*.

In particular, consider the *universal plurality*, that is, the plurality of every object there is. On the uncontroversial assumption that there are two or more objects, the corresponding instance of Plural Cantor can be formulated as follows:

> Plural Profusion
>
> There are more pluralities than objects.

This means that there can be no injective mapping of pluralities into objects.

The problem is that this technical result clashes with a wide range of views in metaphysics, philosophy of mathematics, and semantics. To get the problem in focus, consider first all the stars in the universe, *ss*. There appear to be many examples of injective mappings from subpluralities of *ss* into objects. Consider the operator 'the set of'. According to its typical usage, this operator defines an injective mapping from subpluralities of *ss* to objects. If *xx* and *yy* are distinct, so are their object-level correlates, that is, the set of *xx* and the set of *yy*. (For a detailed discussion see Linnebo 2010.) Metaphysics provides other important examples. An interesting case is that of propositions or facts (see McGee and Rayo 2000). It is widely assumed that if *xx* and *yy* are distinct, then the propositions or facts expressed by '*xx* exist' and '*yy* exist' are also distinct. Thus, the functional expressions 'the proposition that . . . exist' and 'the fact that . . . exist' define injective mappings from subpluralities of *ss* to objects. However, if these functional expressions were applicable not just to subpluralities of *ss* but to all pluralities, we would run into trouble. Suppose that, for any things *xx* whatsoever, there is the set of *xx* (or the proposition that *xx* exist, or the corresponding fact). Then we would have an injective mapping *from pluralities to objects*—in violation of Plural Profusion.

Let a *singularization* be an injective mapping from the subpluralities of some objects *xx* into objects.[2] The plural version of Cantor's theorem constrains what singularizations are possible. For the theorem says that, provided that *xx* have two or more members, there can be no injective mapping of the subpluralities of *xx into these very objects*. That is, if there is a singularization of all the subpluralities of *xx*, then the values of the subpluralities under this singularization cannot all be among *xx* but must "overflow" this plurality. Of course, in some of the examples mentioned above, this is precisely what one would expect. No one expected a set of stars to be itself a star, and likewise for facts, properties, and propositions concerned with stars. The problem arises when this overflowing is impossible, as in the case of the universal plurality. This implies that there can be no singularization of its subpluralities. Any such singularization would have to overflow the universal plurality, which by its assumed universality is impossible.

This is a puzzling result. In our first example, it seemed completely incidental that we considered the subpluralities of all the stars, as opposed to the subpluralities of some other lot of objects. It is not as if stars are more amenable to figuring as elements of sets than any other objects, or, for that matter, to be involved in propositions, facts, or properties. One might therefore have thought that these are *universal* singularizations, in the sense that they are available for any plurality whatsoever.[3] Plural Profusion seems to show, in one fell swoop, that none of the mentioned singularizations, nor any other, can be universal: there simply aren't enough objects to enable a singularization of absolutely all pluralities.

[2] As before, this talk of mappings can either be taken as primitive or be understood as shorthand for claims that officially talk merely about pluralities of ordered pairs (see Appendix 3.A). For ease of communication, we will mostly indulge in talk about mappings, which could always be translated into talk about pluralities of pairs.

[3] This is also a consequence of the liberal view of definitions canvassed in Section 4.4 and developed in more detail below.

A common reaction in the literature has been to take this result at face value as a surprising limit on what singularizations there can be.[4] This reaction is not without problems, however. Singularization seems to play an important role in natural language and in a wide range of theoretical contexts, from mathematics to semantics. What are we to say about all these apparent singularizations? Since it is not an option to reject singularizations altogether, the most promising response is to find a way to restrict their availability. We can allow these singularizations to be undefined on certain pluralities or lift the requirement that the associated mappings always be injective. However, this "compromising" response faces a threat of arbitrariness. Restricting the scope of a device of singularization raises the question of whether the restriction is adequately motivated.

We have ended up in an awkward position. Plural Cantor seems to show that there can be no universal singularization, and as we have just seen, this threatens to introduce some arbitrary and unmotivated restriction on what singularizations there can be. Let us therefore reexamine the argument. Might there be some way to reconcile Plural Cantor with the availability of universal singularization? Given Plural Cantor, we know that any singularization of the subpluralities of some things would have to overflow these things. So a reconciliation would have to find a way to permit this kind of overflow without exception. There have been two attempts to permit this.

The better known strategy is *generality relativism*, which denies that absolute generality is possible. This view entails that no plurality is universal in an absolute sense. The most we can ever have is a relative kind of universality, which can always be surpassed. Although a plurality xx may be universal with respect to our current interpretation of the quantifiers—that is, $\forall x(x \prec xx)$—it is possible to find an extended interpretation with respect to which xx is no longer universal—that is, $\exists^+ x \neg(x \prec xx)$ (where the $\exists^+$ indicates that the quantifier is taken in this extended sense). This yields an operation that extends any given interpretation I to a strictly more inclusive interpretation I^+. On the resulting view, any plurality—including one that is universal with respect to our current interpretation of the quantifiers—can be surpassed once we adopt a more inclusive interpretation of the quantifiers. Clearly, this view makes the world safe for singularization. Since no

[4] See, e.g., McGee and Rayo 2000, Rayo 2002, and Uzquiano 2015a. Note also that this attitude towards singularization is implicit in much of the philosophical literature on plural logic. An analogous point is true with respect to the parallel case of nominalization and higher-order logic.

plurality is universal in an absolute sense, there is no obstacle to unrestricted singularization. Provided that singularization leads to an expansion of the interpretation of the quantifiers, we can safely accept its effect of always surpassing any plurality with which we begin.

11.3 A trilemma

There is another strategy for blocking the argument against universal singularizations, namely to restrict the axiom scheme of plural comprehension. This strategy has no truck with generality relativism and accepts that an absolute interpretation of the quantifiers is possible. Nor does the strategy have any quarrel with Plural Cantor: it is perfectly true, for any plurality *xx* with two or more members, that the subpluralities of *xx* are strictly more numerous than the members of *xx*. Rather, unlike the generality relativist, who seeks to retain traditional plural logic, the strategy in question challenges our naive assumptions concerning what pluralities there are. After all, in the argument above, trouble arose only when we assumed that there is a universal plurality, which enabled us to derive the problematic instance of Plural Cantor, namely Plural Profusion. (Of course, we would get the same effect from any other plurality that is too large to allow of singularization, as its correspondingly small complement cannot accommodate the overflow that would result.)

Needless to say, the big challenge for this strategy is to explain why there are no pluralities that are so large that they cannot be singularized. The existence of such pluralities is underwritten by the unrestricted plural comprehension scheme of traditional plural logic. Any rejection of the currently accepted version of plural logic will of course have to be well motivated. Attempts to provide such a motivation have in fact been made, targeting the unrestricted plural comprehension scheme in particular. A promising idea derives from the thought that domains of quantifications might be *extensionally indefinite*, or not properly circumscribed, while every plurality is extensionally definite. The idea is nicely summarized in the following passage by Stephen Yablo:

> The condition $\phi(u)$ that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate question whether x satisfies $\phi(u)$ or not. How then can it fail to be a determinate matter what are *all* the things that satisfy $\phi(u)$? I see only one answer to this. Determinacy of the ϕ's follows from

(i) determinacy of $\phi(u)$ in connection with particular candidates,
(ii) determinacy of the pool of candidates.

> If the difficulty is not with (i), it must be with (ii). (Yablo 2006: 151–2; some notation and terminology has been modified)

Perhaps the price of absolute generality is that the range of our quantifiers becomes extensionally indefinite (or "indeterminate", as Yablo might put it). Since pluralities are extensionally definite, however, this would give us a reason to restrict plural comprehension so as to reject the universal plurality. Of course, the main challenge for this approach will be to articulate the notion of extensional definiteness and show that it has the right properties. This task was begun in the previous chapter and will be completed in next.

Let us take stock. Assume that absolute generality is possible and there is a plurality that is universal in this absolute sense. Then Plural Profusion entails that there cannot be a universal singularization. For if there were, such a singularization would yield an injective mapping from subpluralities of the universal plurality to objects, contradicting Plural Profusion. Moreover, since this argument can be given with the quantifiers interpreted absolutely, which is assumed to be possible, it is not an option to object that the argument equivocates by expanding the interpretation of the quantifiers somewhere along the way. So we have a trilemma. We must accept one of the following three horns.

First horn

Universal singularizations are impossible.

Second horn

It is impossible to quantify over absolutely everything.

Third horn

There is no plurality that is universal or all-encompassing.

The trilemma confronts us with a difficult theoretical choice.[5] As we have seen, there are several examples of devices of singularization in natural

[5] Might a fourth option be possible, namely to challenge Plural Cantor? Since this is a mathematical theorem, it can be no more controversial than the assumptions on which it rests. The only point of attack seems to be the impredicative plural comprehension axiom that is involved in some versions of the theorem. We don't find this challenge at all promising, for two reasons. First, some versions of the theorem require only predicative comprehension, as we saw in Appendix 3.A. Second, even for the versions that require impredicative comprehension the requisite impredicative plural comprehension was defended in Appendix 10.A.

language and some of them appear to be fully general. Next, there is a wealth of examples of assertions that seem to be about absolutely everything. Finally, the existence of a universal plurality is underpinned by the principle of plural comprehension enshrined in the traditional version of plural logic and thus appears to stand on solid ground. How should the trilemma be resolved? In what follows, we shall assess each of its three horns.

11.4 Relativism and inexpressibility

It turns out that the rejection of absolute generality is fraught with difficulties. Let us mention three problems. One is simply that absolute generality very much appears to be possible, for instance when we truly assert that the empty set has absolutely no elements. It would take a very good reason to go against such a robust appearance. A second problem is that absolute generality is needed in order to express various general views that we find interesting, such as the physicalist claim that absolutely everything is physical. To disallow the expression of these views would be to disallow a lot of potentially fruitful theorizing. As Williamson remarks, "[i]f the unexamined life is not worth living, the credentials of a life without absolutely general thought are shaky" (2003, 452). We take this to be a very serious complaint.

The most intriguing argument against generality relativism, however, is that the view cannot coherently be expressed. David Lewis states the point with characteristic verve:

> Maybe the singularist replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it! (Lewis 1991, 68)

Some unpacking may help. Consider the claim that my current language does not quantify over absolutely everything. This entails that there is something over which my quantifiers do not range. But this is incoherent, as I am now using a quantifier to assert the existence of something not in the range of my quantifiers.

A relativist might attempt to do better by expressing her view as a claim about interpretations, namely that for every interpretation of our quantifiers, there is a more expansive interpretation. We use subscripts to indicate the interpretation given to the quantifiers. Let $I \subset J$ abbreviate $\exists_J x \forall_I y(x \neq y)$. One may then attempt to express relativism as follows:

$$\forall I\, \exists J (I \subset J) \tag{11.3}$$

Promising though it may be, the attempt fails, as shown by the following dilemma. Assume first that the quantifiers '$\forall I$' and '$\exists J$' in (11.3) range over absolutely all interpretations. Then (11.3) expresses what it is meant to express. But in so expressing it, one is violating the view expressed. For just as there are arguments that it is impossible to quantify over all ordinal numbers or all sets (see Florio 2014b, Section 3.1), there are analogous arguments concerning quantification over interpretations. Alternatively, assume that the quantifiers in (11.3) do not range over absolutely all interpretations.[6] Thus understood, (11.3) is compatible with the view it is meant to express. The problem is now that (11.3) fails to express the view properly. All that is expressed is that every interpretation *in some limited range of interpretations* can be extended. But this is compatible with there being a maximal interpretation outside of this limited range.

The standard response by generality relativists, advocated for instance in Glanzberg 2004 and Parsons 2006, is to invoke *schematic generality*. This idea traces back to Russell's use of free variables to achieve a form of generality that goes beyond that afforded by the quantifiers.i

> For our purposes [the distinction between 'all' and 'any'] has a different utility, which is very great. In the case of such variables as propositions or properties, 'any value' is legitimate, though 'all values' is not. Thus we may say: '*p* is true or false, where *p* is any proposition', though we can not say 'all propositions are true or false'. (Russell 1908, 229–30)

In effect, we use free variables to achieve a version of absolutely general universal quantification. Consider an operation which, when applied to any interpretation I yields an extended interpretation I^{+}; an example is the operation described in Section 11.2. Using free variables, relativism can now be expressed schematically as follows:

$$I \subset I^{+} \tag{11.4}$$

The use of schematic generality is severely limited, however. Schematic statements cannot be negated and cannot be freely combined in other truth-functional ways. Consider for instance the negation of (11.4) and

[6] See, for instance, the relativist argument developed (as a foil) in Williamson 2003, Section IV.

read this schematically. The formula would express a universally generalized negation, not the desired negated universal generalization. More generally, schematic generality enables us to express absolutely general Π_1-sentences, but not Σ_1-sentences or beyond.[7] We find this expressive limitation hard to accept. Anything that can be expressed can also be denied.

Can we do better by exploiting alternative expressive resources? An interesting option is to formalize reasoning about expansions of quantifier interpretations by means of modalities.[8] So (11.3) receives a modal reading:

(11.5) Necessarily, for any interpretation *I*, there could be an extended one *J*

or in symbols:

(11.6) $\Box\forall I \Diamond\exists J\,(I \subset J)$

How should the modal operators be interpreted? The ordinary metaphysical interpretation is problematic. For the existence of the relevant objects, such as pure sets, is often assumed to be metaphysically necessary, which rules out any variation of the domain of such objects across metaphysical possibilities. Some writers favor an interpretational understanding of the modality, where the modal operators enable us to theorize about the result of certain changes to the interpretation of the language.[9]

Suppose this understanding of the modality can be made out. What would have been achieved? A desire for greater expressive adequacy led to the adoption of resources that allow us to retrieve, or at least to simulate, full absolute generality. For the strings '$\Box\forall$' and '$\Diamond\exists$' can now be used as devices of generalization: not just over everything in the range of the quantifiers as *currently* interpreted, but over everything in their range on *any possible* interpretation. Indeed, the "mirroring theorem" of Linnebo 2010 shows that, under plausible assumptions, the "modalized quantifiers" '$\Box\forall$' and '$\Diamond\exists$' behave precisely like ordinary quantifiers as far as logic is concerned. So these can be seen as ways to recover a form of absolute generality from within a theoretical standpoint that shares many of the motivations of relativism.

Where does this leave us? We set out to develop a form of relativism. We ended up defending a form of absolute generality—albeit with an uncon-

[7] We can, however, use operators or function symbols to make *some* existence claims even in the scope of schematic generalization. An example is (11.4) where '+' represents an operation we can apply to any given interpretation *I* so as to form an extended interpretation I^+.

[8] See Fine 2006, Linnebo 2010, and Studd 2013.

[9] See Fine 2006; Linnebo 2018, Sections 3.5–3.6; Studd 2019, Sections 4.4 and 6.1.

ventional understanding of absolute generality. This unconventional feature emerges particularly clearly in connection with the following modalized analogue of the ordinary plural comprehension scheme:

$$\Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \varphi(y))$$

Recall from Chapter 10 that every plurality is rigid: it has the same members at every world at which it exists. This entails that the above scheme is invalid. For example, let $\varphi(y)$ be the condition '$y = y$'. Since the domain can vary from possible world to possible world, so can the extension of this condition. By contrast, a plurality cannot vary in membership. It is therefore impossible for there to be a plurality that is necessarily coextensive with this condition. The upshot is that, when the strings '$\Box\forall$' and '$\Diamond\exists$' are used to recover absolute generality, the plural comprehension scheme, couched in terms of this form of generality, needs to be restricted. In short, an attempt to defend the first horn of our trilemma has morphed into a view that is more usefully regarded as a defense of the third horn. Specifically, we have used pluralities, which are tracked rigidly across possible worlds, to explicate the notion of an extensionally definite collection, or Yablo's corresponding notion of a "determinate pool of objects".

For the purposes of this book, we prefer a more direct approach. Instead of adding modal operators to shore up a version of the first horn, we would like to develop the third horn directly—which we do in the next chapter.

11.5 Traditional absolutism and ascent

In the remainder of this chapter, we will explore the second horn of our trilemma. This horn concedes that there cannot be any universal singularizations, while retaining absolute generality over an extensionally definite domain, represented by a universal plurality. Throughout this discussion, traditional plural logic will therefore be assumed. We call the resulting view *traditional generality absolutism*.

One of the main challenges confronting this view is to develop a model theory for a language whose quantifiers are interpreted absolutely. The usual set-based model theory is obviously unavailable, since the domain now consists of absolutely everything there is and there is no universal set according to standard set theory (see Section 7.7). How, then, should advocates of absolute generality represent the domain of their absolute quantifiers and the semantic values of the predicates defined on this all-inclusive domain?

An answer that has recently gained a lot of support is that the model theory for a first-order language with absolute generality can and must be given in a plural or higher-order metalanguage.[10] In this metalanguage, we let domains be pluralities or concepts. The domain of a language with absolute generality will then correspond to the universal plurality or a universal concept. This way to talk and reason about domains requires no singularization whatsoever. Although we talk informally about "the domain", using a singular definite description, we officially have in mind the many objects, or the many instances of a concept, over which the quantifiers range. A similar strategy allows us to ascribe semantic values to predicates. Although we informally talk about "the semantic value" of a predicate, officially there are many objects or a concept representing the predicate's semantic contribution. In short, in order to develop the model theory for a first-order language with absolute generality, we must ascend to a language with plural or second-order resources.

Our discussion in Section 7.5 showed another instance of this phenomenon. A model theory for PFO+ with absolutely unrestricted quantification can only be given in a language with another layer of quantification, such as superplural quantification or quantification over plural concepts. In fact, this ascent phenomenon can be shown to continue further, as we will now explain in detail. We will focus on the plural hierarchy, although it is not difficult to adapt our discussion to the corresponding conceptual hierarchy. For the two hierarchies have a common type-theoretic structure. So to emphasize the parallel between them, we will often speak of a *type-theoretic* hierarchy. Recall our terminology when the types receive a plural interpretation: a language of order 1 is just a regular first-order language, while order 2 adds plural quantification, order 3 adds superplural quantification, and so on. Thus, a language of order $n + 1$ quantifies over what we call pluralities of level n.

A more detailed argument for the ascent can now be set out as follows.

Premise 1

Traditional plural logic is valid.

Premise 2

Absolute generality is possible at every order of the hierarchy; that is, for every order, it is possible to quantify over absolutely all entities at that order.

[10] See, e.g., Rayo and Uzquiano 1999, Rayo and Williamson 2003, and Williamson 2003.

To formulate the third and final premise, let a *generalized semantics* be a theory of all possible interpretations that a language might take, without any artificial restrictions on the domains, interpretations, and variable assignments. A generalized semantics is thus an instance of model theory, in our liberal sense of that term. The premise can now be stated as follows.

Premise 3 (semantic optimism)

Given any legitimate language, it should be possible to develop a generalized semantics for that language.

Finally, we have the following theorem.

Ascent theorem (basic form)

Assume traditional plural logic and the possibility of absolute generality. Then a generalized semantics for a first-order language cannot be given in another first-order language but can be given in a language with plural quantification.

The result of the three premises and the theorem is that we are pushed from a first-order to a plural language.

The question now arises: what about the semantics of a language with absolutely general plural quantification? It turns out that the considerations that require the initial ascent from a language of order 1 to a language of order 2 require further ascents as well. For we have the following:

Ascent theorem (arbitrary finite form)

Assume traditional plural logic and the possibility of absolute generality at every finite order $n \geq 1$. Then a generalized semantics for a language of order n cannot be given in another language of order n but can be given in a language of order $n + 1$.

So at every finite order, the desire for a generalized semantics pushes us one step up. This results in an ascent up through all the finite orders.[11]

[11] More fine-grained results are possible as well. The Ascent Theorem applies to languages that are sometimes called *full* (or *plenary*) in the sense that they contain predicates whose arguments can be variables of order n, where n is the order of the language. If a language of order n is not full, the formulation of a generalized semantics for it requires only that we ascend to a *full* language of order n. This is why, for example, the plurality-based model theory for PFO was carried out in PFO+. For a summary of these results and references to the literature, see Florio 2014b, Section 4.1.

In fact, as shown by Linnebo and Rayo (2012), an additional principle, broadly in the spirit of those behind the theorems, extends the hierarchy of higher-order languages into the transfinite. The additional principle states, roughly, that for any collection of languages in the hiearchy, we can form a "union" language that encompasses all the languages in the collection.[12] Linnebo and Rayo (2012) prove a generalization of the ascent theorem into the transfinite.

> ASCENT THEOREM (TRANSFINITE FORM)
>
> Assume traditional plural logic and the possibility of absolute generality at every order. Then we cannot develop a generalized semantics for a language of order α in another language of order α. But for every successor ordinal α, we can develop a generalized semantics for a language of order α in a language of order $\alpha + 1$.[13]

In fact, there is reason to think that Tarski knew all of this (and more):

> [T]he setting up of a correct definition of truth for languages of infinite order would in principle be possible provided we had at our disposal in the metalanguage expressions of higher order than all variables of the language investigated. (Tarski 1935, 72)

The end result is that defenders of the new orthodoxy, seeking to secure the possibility of absolute generality while holding on to semantic optimism, are pushed by the ascent theorems higher and higher up through the orders of the hierarchy.

The only way to stop this ascent would be to give up on semantic optimism, which insists on a generalized semantics. Why insist on this? As remarked, a generalized semantics is just a higher-order version of ordinary set-based model theory. It is therefore natural to think that a generalized semantics is required in order to give an appropriate definition of logical

[12] To avoid inconsistency, we must restrict the collections to which the principle applies. If languages are indexed by ordinals, a plausible restriction is to any bounded collection of languages: for a limit ordinal λ, if one is prepared to countenance languages of order β for every $\beta < \lambda$, then one should also countenance a language of order λ. Without this restriction, we would be able to prove an inconsistency. See Florio and Shapiro 2014 and Linnebo and Rayo 2014 for further discussion.

[13] What about a language whose order is an infinite limit ordinal λ? Since this language is contained in the language of order $\lambda + 1$, the theorem ensures that its generalized semantics can be developed in a language of order $\lambda + 2$.

consequence, just like Tarski's notion of logical consequence requires model theory.

One might try to resist this natural thought by recalling an observation made in Section 7.4. By appealing to an appropriate set-theoretic reflection principle, we can ensure that the standard definition of logical consequence as truth preservation in every set-based model is extensionally equivalent to the definition of logical consequence as truth preservation in every plurality-based model. Does this show that a generalized semantics isn't required after all? We don't think so. As indicated in Section 7.9, one response is that we would like a theory of logical consequence that is not only extensionally but also intensionally correct. For instance, by the downwards Löwenheim-Skolem theorem, we know that it would be extensionally correct to define logical consequence by quantifying solely over finite and countably infinite models. Although extensionally correct, this definition of logical consequence would be inferior to Tarski's, because it badly fails to capture the intended intension. An analogous argument can be made for considering not only set-based models but all models, as is done in generalized semantics. Moreover, the ability to capture the intended intension, which here requires quantification over all models, is essential when we lack independent means of determining the correct extension. In that case, it is only against the backdrop of an intensionally correct theory that we can check whether any other theory is in fact extensionally adequate.

Another, more direct response is that generalized semantics is legitimate and interesting in its own right, irrespective of its contribution to theorizing about logical consequence. Our language has one interpretation. But there are myriad other interpretations that it might have had. It is a legitimate undertaking to study all these interpretations and how the truth of sentences is affected by the choice of interpretation.

11.6 Ascent and inexpressibility

Let us therefore accept that traditional generality absolutism pushes us up through the orders of the hierarchy and proceed to inquire about the significance of this ascent phenomenon. It turns out that the ascent gives rise to three complaints that mirror those we leveled against relativism in Section 11.4.

First, type-unrestricted generality appears possible. For example, it appears meaningful to ask whether the law of extensionality holds at every order of the type-theoretic hierarchy. It would take a very good reason to go against such a robust appearance. Yet no such generalization is available on the type-theoretic view. While quantification of any specific order is available, there is no such thing as quantification across all orders at once.

Second, type-unrestricted generality is needed to engage in many interesting and potentially valuable forms of theorizing. We already mentioned the question of extensionality, which figures in some theoretically important claims, for example, that extensionality holds at every order of the plural hierarchy but not of the conceptual hierarchy. Likewise, it is an important insight, which deserves to be properly expressed, that a version of Cantor's theorem holds at every order of the type-theoretic hierarchy. Coupled with widely held assumptions about plural comprehension, this means that there are more pluralities than objects, more superpluralities than pluralities, and so on up through all the possible levels. For a final example, consider the claim that the principle of compositionality holds at every order, that is, that at every order, the semantic value of a complex expression is determined as a function of the semantic values of the expression's simpler constituents.[14] None of these questions can be properly expressed and discussed in the type-theoretic setting. We thus seem to be confronted with examples of expressive limitations that curtail certain forms of systematic and valuable theorizing.

In fact, the view that type theory suffers from expressive limitations has a long history. Wittgenstein alludes to it in the *Tractatus* (Proposition 4.1241) and formulates it explicitly in his pre-Tractarian period:

> Types can never be distinguished form each other by saying (as is often done) that one has these *but* the other has those properties, for this presupposes that there is a *meaning* in asserting all these properties of both types. (Wittgenstein 1979, 106)

For essentially these reasons, he concludes two pages later that "a THEORY of types is impossible". Very similar considerations are echoed by Gödel twenty years later:

> The theory of simple types [...] has the consequence that the objects are divided into mutually exclusive ranges of significance, [...] and that therefore each concept is significant only for arguments belonging to one of these ranges, i.e., for an infinitely small portion of all objects. What makes

[14] See Linnebo 2006 for some further examples.

> the above principle particularly suspect, however, is that its very assumption makes its formulation as a meaningful proposition impossible [...]. Another consequence is that the fact that an object x is (or is not) of a given type also cannot be expressed by a meaningful proposition. (1944, 466)

It might be objected that our examples of expressive limitations are biased.[15] From our point of view, there are indeed important generalizations that the type theorist cannot express. But from the type theorist's point of view, the alleged examples of inexpressible insights can be dismissed as ungrammatical gibberish. This is a perceptive and interesting complaint, which leaves us in a difficult dialectical situation. From one point of view, there is evidence against the opposing view. From the opposing point of view, this alleged evidence isn't even meaningful!

How can we get beyond this apparent impasse? It is true that the type theorists can stubbornly reject the attempted examples of expressive limitations without any fear of thereby contradicting themselves. But we claim it would be bad methodology to do so. Greater expressive power appears possible; there are consistent ways to develop this greater expressibility; and the greater expressibility promises to be theoretically useful.[16] In such cases, we contend, it is good methodology to press ahead, despite the protestations of the coherent naysayers—though obviously with the epistemic caution that behooves every exploration of an unconventional hypothesis.

Third, an objection to the semantic ascent through the type-theoretic hierarchy is that we are precluded from properly stating the type theorists' view that there is a hierarchy strictly divided into levels and without a top level. This is analogous to the case of relativism. Recall the relativist's predicament: to state that every quantifier interpretation can be extended, we need to avail ourselves of absolute quantification over interpretations. Likewise, to state that quantification of every order can be extended by quantification of some even higher order, we need to generalize across all the orders simultaneously. The hierarchy has no maximal level, yet we are precluded from properly expressing that.

Might the expressive limitations be overcome by appealing to schematic generality, as discussed in Section 11.4? In this case, the schematic generality would reside in the type indices. Where τ is a type, a claim $\varphi(x^\tau)$ would be

[15] A version of this objection is discussed in Krämer 2014.
[16] For the second point, see Section 11.7.

understood as conveying that the claim holds for any type τ.[17] However, as noted in our discussion of generality relativism, the logical complexity of the generalizations that can be captured by schematic generality is extremely limited.

In sum, generality relativism and traditional generality absolutism have far more in common than has been acknowledged in the existing literature: both suffer from expressibility problems. We have discussed three such problems: the apparent meaningfulness of certain absolute generalizations, their potential theoretical utility, and the inability to properly express one's own view without access to such absolute generalizations. Our discussion motivates taking a closer look at the third alternative, namely absolute generality over a domain that is extensionally *in*definite, or not properly circumscribed. This is the task of the next and final chapter.

Before turning to this task, however, we would like examine a strategy that might allow the traditional absolutist to restore full expressibility. We will find that, while promising, this strategy ends up transforming traditional absolutism into a view that has much in common with the third alternative.

11.7 Lifting the veil of type distinctions

As we have seen, semantic considerations push the traditional absolutist higher and higher up through the type-theoretic hierarchy. But this ascent phenomenon leads to expressibility problems. We now explore another perspective on the debate. Consider the entire plural hierarchy to which the traditional absolutist ends up committed. At the bottom, there is an extensionally definite domain of individuals, which make up level 0. Then there is level 1, which adds pluralities; level 2, which adds superpluralities; and so on. Let the traditional absolutist make her choice about how high to go. Our only assumption is that there is no maximal level of the hierarchy. That is, for every level α in the plural hierarchy, there is also level $\alpha + 1$. These levels are reflected in the type distinctions of our language: variables of each type take their values exclusively from the corresponding level.

What happens if we abandon these type distinctions and bring all the different sorts together? Doing so would be a radical change in perspective. We would, as it were, lift the veil of type distinctions and thus gain a new perspective on reality. We will now defend the coherence of this new

[17] Russell famously exploits this kind of "typical ambiguity"; see Russell 1908, 251.

perspective. We will also show that, from the new perspective, traditional plural logic is no longer valid.[18]

As a warm-up case, imagine a community of extreme Cartesian dualists whose language involves a strict type distinction between mental and physical vocabulary. Members of this community regard the application of mental predicates to physical terms as meaningless rather than false. Likewise, they regard the application of physical predicates to mental terms as meaningless rather than false. This prevents them from being able to generalize at once over both the mental and the physical domain. For example, these dualists cannot express claims such as:

(11.7) Everything is either mental or physical.

(11.8) Nothing is both mental and physical.

The situation is analogous to the one described in the previous section. Like the type theorist we encountered there, the dualists can dismiss these alleged examples of inexpressible claims as ungrammatical gibberish. How can we convince them to abandon their type distinction and adopt a perspective that permits the expression of the above claims? From our point of view, the dualists' type distinction is dogmatic and parochial. But this charge is supported by evidence which, from their point of view, isn't even meaningful! We face an impasse, which again can only be overcome by showing to the dualists the methodological flaws of their dogmatism. Greater expressive power appears possible, there are consistent ways to develop this greater expressibility, and the greater expressibility promises to be theoretically useful. As before, we contend that it is good methodology in such cases to explore the expressively richer perspective. So let us describe a suitable language in which that can be done.

The language of the community of extreme Cartesian dualists, we recall, has distinct sorts for mental and physical vocabulary. We translate this language into a one-sorted language in which all syntactic restrictions based on the two sorts have been removed. We add two new predicates 'M' and 'P' for being mental and being physical, respectively. Using these predicates together with the "all-purpose" variables of the one-sorted language,

[18] Simons (2016) and Oliver and Smiley (2016, Chapter 15) share our aim of developing a logic of higher-level pluralities in an untyped language and discusses some axioms that might be appropriate for this logic. Unlike them, however, we pursue this aim indirectly by first formulating a typed logic of higher plurals and then translating it into an untyped system.

we can track the dualists' sortal distinction and interpret it as a form of quantificational restriction. When they assert, relative to the mental sort, that everything is F, we interpret them as asserting that every M is F. Likewise, when they assert, relative to the physical sort, that something is G, we interpret them as asserting that some P is G. By means of this translation, we regain full expressibility. For example, we can now state the claims that the dualists were unable to express:

(11.9) $\forall x(Mx \vee Px)$

(11.10) $\neg\exists x(Mx \wedge Px)$

Let us return to the typed language that is our real concern, namely the language of the plural hierarchy—call it $\mathcal{L}_1$. Proceeding as in our warm-up case, let us bring its many sorts together by translating this language into a standard one-sorted language $\mathcal{L}_2$. We want to capture the sortal distinctions of $\mathcal{L}_1$ in the one-sorted setting of $\mathcal{L}_2$. To this end, we let $\mathcal{L}_2$ contain a new two-place predicate 'L' for the level of a plurality. Intuitively, '$L(x, 0)$' means that x is an individual; '$L(x, 1)$', that x is a plurality; '$L(x, 2)$', that x is a superplurality; and so on. (We are assuming the language can quantify over enough ordinal numbers to index all the levels of the plural hierarchy.)

Let us describe a translation τ from $\mathcal{L}_1$ to $\mathcal{L}_2$. Every atomic formula containing no plural vocabulary is translated as itself. An atomic formula containing plural vocabulary is translated by replacing each plural expression with a singular counterpart. We avoid clashes of terminology by ensuring that the translation of singular and plural vocabulary does not overlap. Moreover, we reserve the special symbol 'η' for a *membership* relation that translates plural membership. Here are some examples:

$$\begin{array}{rcl} Fa & \stackrel{\tau}{\longmapsto} & Fa \\ Gxx_i & \stackrel{\tau}{\longmapsto} & Gx_i \\ R(xx_i, y_j) & \stackrel{\tau}{\longmapsto} & R(x_i, y_j) \\ y_j \prec xx_i & \stackrel{\tau}{\longmapsto} & y_j \; \eta \; x_i \end{array}$$

The translation commutes with the logical connectives:

$$\begin{array}{rcl} \neg\varphi & \stackrel{\tau}{\longmapsto} & \neg\tau(\varphi) \\ \varphi \wedge \psi & \stackrel{\tau}{\longmapsto} & \tau(\varphi) \wedge \tau(\psi) \end{array}$$

Finally, we come to the action of τ on the quantifiers. Singular quantifiers retain their sort, since this is the only sort available in $\mathcal{L}_2$, but we restrict them by means of the formula '$L(x, 0)$'. Plural quantifiers are translated as singular quantifiers restricted by means of the formula '$L(x, 1)$'; superplural quantifiers, by means of '$L(x, 2)$'; and so on. So we have:

$$\begin{array}{rcl} \forall y_j\ \varphi & \stackrel{\tau}{\longmapsto} & \forall y_j(L(y_j, 0) \to \tau(\varphi)) \\ \forall xx_i\ \varphi & \stackrel{\tau}{\longmapsto} & \forall x_i(L(x_i, 1) \to \tau(\varphi)) \\ \forall xxx_i\ \varphi & \stackrel{\tau}{\longmapsto} & \forall x_i(L(x_i, 2) \to \tau(\varphi)) \end{array}$$

In a nutshell, a speaker of $\mathcal{L}_1$ is interpreted as a speaker of an untyped language whose sortal distinction is merely the syntactic expression of a quantificational restriction.

The kind of translation we are proposing is not entirely unprecedented. (See also Quine 1956.) Suppose $\mathcal{L}_1$ is the language of PFO+, which has only two types: singular and plural. Then $\mathcal{L}_2$ is essentially the kind of one-sorted version of plural logic that we discussed in Section 5.3.[19] Returning to the general case, we may think of $\mathcal{L}_2$ as a generalization of this one-sorted plural language. The single sort of variables of $\mathcal{L}_2$ permits different forms of reference: singular, plural, superplurals, and so on. That is, we have a single sort of "all-purpose" variables whose possible assignments include those of all the different types of variable that are available in $\mathcal{L}_1$.

Is this one-sorted language $\mathcal{L}_2$, and the translation into it, permissible? Let us begin by examining whether we run a risk of inconsistency by translating in this way. We need to examine how theories formulated in $\mathcal{L}_1$ relate to their translations. Let T be an $\mathcal{L}_1$-theory and let T^* be an $\mathcal{L}_2$-theory whose axioms are the translations of the axioms of T. Then, we have the following key fact.[20]

Fact 11.1 T is consistent if and only if T^* is consistent.

Thus, formal consistency is not a concern when we lift the veil of type distinctions.

We therefore turn to more philosophical issues concerning the language $\mathcal{L}_2$. Should its single sort of terms be taken to refer to sets? If so, the traditional absolutist might protest that the translation involves a change of

[19] The only difference is the unimportant one that $\mathcal{L}_2$ has a predicate 'η' for membership, whereas the other language has a predicate '$\preccurlyeq$' for being among. As we have seen, however, these two predicates are interdefinable.

[20] See Enderton 2001, 300, Theorem 44A.

subject. The plural variables of $\mathcal{L}_1$ are used to refer plurally, not to effect singular reference to sets; so to be adequate, a translation needs to respect that fact. But in fact, it is neither obligatory nor particularly natural to take the single sort of terms of $\mathcal{L}_2$ to effect singular reference to sets. Many theorists take sets to lack spatiotemporal location, and almost all take them to lack causal powers. But pluralities of every level can have location, time, and causal powers; for example, some children may be located in the garden, break a window, and so on.

If not to sets, to what *do* the variables of $\mathcal{L}_2$ refer? As mentioned, our proposal is that each of these variables is capable of a variety of different forms of reference: singular, plural, superplural, and so on. The assignment to each such variable will be made in some metalanguage by means of another variable with the same capabilities concerning its forms of reference. This view isn't objectionable to traditional absolutists in the way it would be objectionable to interpret the terms of $\mathcal{L}_2$ as referring to sets. True, $\mathcal{L}_2$ lifts the veil of the syntactic type distinctions found in $\mathcal{L}_1$. But after lifting the veil, each term retains precisely the form of reference it had before.[21]

So far, we have acquitted $\mathcal{L}_2$ of the charges of risking inconsistency and of changing the subject by translating terms that refer plurally as terms that refer to sets. What positive reason might we have to accept $\mathcal{L}_2$? Our answer is that in $\mathcal{L}_2$ we can express everything we wanted to, but couldn't, express in $\mathcal{L}_1$. Here are some examples. First, we can raise the question of cumulativity: can a plurality of level $n + 1$ have members only of level n or also of any level lower than $n + 1$? For example, does 'my children, your children, and Bob' refer to such a mixed-level plurality? Second, what is the relation between a singleton plurality and its single member? Should these be identified or not? For example, do 'the objects identical to Bob' and 'Bob' co-refer? Third, do extensionality principles hold at every level of the plural hierarchy? For example, should we accept an indiscernibility principle (Sections 2.4 and 10.5) governing each level? Based on these considerations, we contend that traditional absolutists have good reason to accept the translation of their plural logic, generalized to pluralities of all levels, into the one-sorted language of higher pluralities, $\mathcal{L}_2$.

We now face a crucial question: can the all-purpose variables of the one-sorted language $\mathcal{L}_2$ be "pluralized"? In other words, can we introduce

[21] The view that all-purpose variables can effect generalized forms of plural references is embraced by some theorists who develop higher-level plural logic directly rather than indirectly by lifting the veil. See footnote 18.

variables that relate to the all-purpose variables the way ordinary plural variables relate to ordinary singular variables? Consider the question from the point of view of our opponent, the traditional absolutist. We are supposing, recall, that her plural language $\mathcal{L}_1$ contains *all* the forms of pluralization that are available. Moreover, pluralization is a relationship that holds between an expression of order $\alpha + 1$ and expressions of order α (or, in the case of cumulativity, orders $\leq \alpha$). Every form of pluralization corresponds to some level of the plural hierarchy associated with $\mathcal{L}_1$. Transposed to the one-sorted setting of $\mathcal{L}_2$, this means that every pluralization of its single sort of variable would have to have values at some level α (or, in the case of cumulativity, at levels $\leq \alpha$). If we are to pluralize the all-purpose variables of the language $\mathcal{L}_2$, it follows that each of the resulting pluralities would be bounded by some level.[22] In particular, there can be no universal plurality with respect to the single sort of variable of $\mathcal{L}_2$—precisely as in the alternative version of plural logic that we will defend in the next chapter.

In short, we have shown how full expressibility can be restored to the traditional absolutist's language while retaining plural reference, superplural reference, and so on. Moreover, we have argued that the traditional absolutist should accept this move. Full expressibility is restored by means of a one-sorted language $\mathcal{L}_2$ that lifts the veil of type distinctions. Crucially, we have found that traditional plural logic is not valid in this new one-sorted setting. When we attempt to pluralize the all-purpose variables of $\mathcal{L}_2$, the interpretation must be confined to some level, and hence some instance of plural comprehension fails—this is the case, for example, for any instance yielding the universal plurality. So, even by her own lights, the traditional absolutist has a reason to countenance an alternative plural logic where plural quantification is bounded by some level. The most plausible development of traditional absolutism thus ends up transforming it into a view that has much in common with our third alternative of developing a critical version of plural logic. In the final chapter, we take a more direct approach to this third alternative.

[22] Oliver and Smiley (2016, Chapter 15) reach the same conclusion via somewhat different reasoning. For us, the boundedness requirement has its root in the typed system and is revealed when the veil is lifted; for them, it is proposed as a natural response to a version of Russell's paradox that would afflict the untyped plural logic if (axioms equivalent to) unrestricted plural comprehension were accepted.

Appendix

11.A The Ascent Theorem

Recall that traditional plural logic assumes unrestricted plural comprehension at every order of the type-theoretic hierarchy. (This gives us, in particular, a universal plurality.) Then, as we saw above, we have the following theorem.

> ASCENT THEOREM (ARBITRARY FINITE FORM)
>
> Assume traditional plural logic and the possibility of absolute generality at every finite order $n \geq 1$. Then a generalized semantics for a language of order n cannot be given in another language of order n but can be given in a language of order $n+1$.

Let us begin by reminding ourselves of the proof of the *basic form* of the theorem, which states that a generalized semantics for a first-order language cannot be given in another first-order language but can be given in a language with plural quantification, such as $\mathcal{L}_{\mathrm{PFO+}}$. First, there is the positive part of the theorem: this was shown in Section 7.3, where we provided a generalized semantics for PFO, and hence for its first-order fragment, in PFO+.

Then, there is the negative part of the theorem. This result relies heavily on the following thesis:

> PLURAL PROFUSION
>
> There are more pluralities than objects.

As we saw in Section 11.2, this thesis follows from Plural Cantor together with the assumption that there is a universal plurality and two or more objects. The negative part of the Ascent Theorem now has a straightforward proof.

Proof. Under the assumption of absolute generality, an ordinary singular predicate can be interpreted by means of any plurality. But by Plural Profusion, there are more pluralities than objects. It follows that interpretations of a first-order language cannot be objects but must be represented by means of higher-order resources. ⊣

We now turn to the proof of the Ascent Theorem in its *arbitrary finite form*. This proof is somewhat involved but can be broken down into three components. First, we need a way of coding ordered pairs of pluralities of arbitrary finite level. Second, we extend the plurality-based model theory to higher level. This is a version of the recursive characterization of truth in a model familiar from Tarski (1935). Finally, taking a cue from Frege (1879) and Dedekind (1888), we show how we can convert a recursive definition to an explicit one by ascending one level in the hierarchy.

Coding of n-tuples of higher-level pluralities

We have shown in Section 7.5 that interpretations and variable assignments can be taken to consist of pluralities of ordered pairs carrying the appropriate semantic information. For instance, an interpretation includes a domain, which is represented by a plurality of pairs of the form $\langle \exists, x \rangle$.

Having defined interpretations and variable assignments, we can talk about the semantic value of an expression E according to an interpretation ii or a variable assignment ss, indicated by $[\![E]\!]_{ii,ss}$. So, for a plural constant tt, we have:

$$\forall x\big(x \prec [\![tt]\!]_{ii,ss} \leftrightarrow \langle tt, x\rangle \prec ii\big)$$

Now we want to generalize these definitions to higher orders. We need some notation for expressions of each finite order. For convenience, we use single lowercase variables for terms and upper case variables for predicates. As usual, the superscript indicates the order of a term. The hierarchy has a plural interpretation but could also be given a conceptual interpretation. We count objects as pluralities of level 0. The following examples illustrate the relation between this notation and the one we have used throughout the book:

$$\begin{array}{ccc} x^0 \prec x^1 & \mapsto & x \prec xx \\ P(x^2) & \mapsto & P(xxx) \end{array}$$

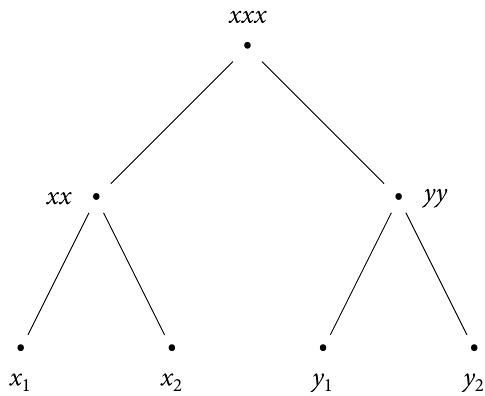
We leave the predicates' arity unmarked. We also suppress the superscript of the symbol '$\prec$' for membership between any two successive levels of the hierarchy. So we write '$x^0 \prec x^1$', '$x^1 \prec x^2$', and so on.

It is essential to have a device for handling higher-level analogues of n-tuples, that is, n-tuples of pluralities of arbitrary (and possibly different) orders. If this can be done, the characterization of an interpretation for a higher-order language will be routine. Thankfully, we have the following theorem:

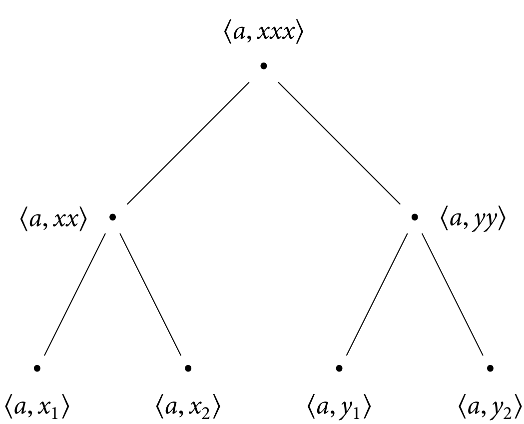
Theorem (*n*-tuples) Assume that for any two objects there is another object that serves as their ordered pair. Given any pluralities $x_1^{k_1}, \ldots, x_n^{k_n}$ whose levels are indicated by the superscripts, we can then code for the ordered n-tuple of these pluralities by means of a single plurality x^k, where k is the maximum level among the k_i.

We will designate this plurality x^k as $\langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle$.

As an initial exercise, consider the case of superplurals. Assume we want to pair an object a with xxx to form $\langle a, xxx \rangle$. As we have seen, xxx is usefully represented in terms of its articulation, for example:



Now, to code the desired ordered pair, all we need to do is add a as a first coordinate to every object that occurs at the base level while retaining the superplural articulation:



This idea, which we have illustrated visually, can now be developed formally and with appropriate generality.

Proof idea.[23] The key step is to define the ordered pair of x^0 and an n-th level plurality x^n. We proceed by induction on n. Assume we have defined $\langle x^0, x^n \rangle$. Then we let $\langle x^0, x^{n+1} \rangle$ be the unique plurality y^{n+1} described by the following equivalence:

$$\forall u^n \big(u^n \prec y^{n+1} \leftrightarrow \exists x^n (u^n = \langle x^0, x^n \rangle \wedge x^n \prec x^{n+1})\big) \tag{11.11}$$

This key step enables us to attach a "tag" x^0 to any higher plurality x^n. And this tagging, in turn, enables us to represent the n-tuple $\langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle$, which can be seen as follows. First, we attach the unique tag i to each $x_i^{k_i}$. Then, we wish to form the union of all of the tagged higher pluralities. Assume for the moment that $k_i = k$ for each i. Then the desired union can be defined as the higher plurality y^k whose members are any z^{k-1} that figures as a member of one of the tagged higher pluralities $x_i^{k_i}$.

We claim that this union y^k represents the desired n-tuple. To establish this claim, we must show how each entry can be retrieved from the union. Suppose we want to retrieve the i-th entry. First, we delete each object at the base level of the union whose tag is distinct from i, while retaining the articulation of the remaining base-level objects. Second, we delete all occurrences of the tag i, again retaining the articulation. This yields $x_i^{k_i}$.

Let us now lift the simplifying assumption that $k_i = k$ for each i and let one of the k_i be less than k. We wish to handle this by raising the level of $x_i^{k_i}$ up to k. We achieve this raising by considering the singleton plurality of $x_i^{k_i}$, and its singleton plurality in turn, and so on until we obtain a higher plurality of level k. We record the number of singleton operations applied by means of a supplementary tag. We now proceed as before but use the resulting plurality instead of $x_i^{k_i}$. When the time comes for retrieving the i-th entry from the union of all the k-th level pluralities, we apply the two steps described in the preceding paragraph and then finish by undoing the j topmost singleton operations, where j is the number recorded by means of the supplementary tag. This yields $x_i^{k_i}$. ⊣

[23] See Linnebo and Rayo 2012, Appendix B, for a detailed proof of their closely related Theorem 1.

Tarski on satisfaction and model theory

We now want to characterize the notion of truth in an interpretation (satisfaction) for a language of order n. As done in Chapter 7, we proceed by first defining the notion of interpretation (as a combination of a domain and an interpretation function) and the notion of variable assignment (and a variant thereof). Then we obtain the definition of truth in an interpretation from the more general relation of truth in an interpretation with respect to a variable assignment, which we characterize recursively. Thus we have generalized the model theory encountered above (Sections 7.3 and 7.5).

Definition (truth in an interpretation) Assume that we have defined an interpretation $i^{n+1} = \langle d^{n+1}, f^{n+1} \rangle$ and a variable assignment s^n. Then we define truth in i^{n+1} with respect to s^n by means of the following clauses.

1. If φ is a formula of the form $P(t_1, \ldots, t_m)$ where P is an m-place predicate and the t_i are of appropriate order (that is, are of order at most n), then:
   $i^{n+1} \vDash \varphi\,[s^n]$ if and only if $\langle [\![t_1]\!]_{i^{n+1},s^n}, \ldots, [\![t_m]\!]_{i^{n+1},s^n} \rangle \prec [\![P]\!]_{i^{n+1},s^n}$
2. If φ is a formula of the form $t^0 = u^0$, then:
   $i^{n+1} \vDash \varphi\,[s^n]$ if and only if $[\![t^0]\!]_{i^{n+1},s^n} = [\![u^0]\!]_{i^{n+1},s^n}$
3. If φ is a formula of the form $t_1 \prec t_2$, where t_1 and t_2 are of the appropriate order (that is, the order of t_1 is strictly below that of t_2), then:
   $i^{n+1} \vDash \varphi\,[s^n]$ if and only if $[\![t_1]\!]_{i^{n+1},s^n} \prec [\![t_2]\!]_{i^{n+1},s^n}$
4. If φ is a formula of the form $\neg\psi$, then:
   $i^{n+1} \vDash \varphi\,[s^n]$ if and only if it is not the case that $i^{n+1} \vDash \psi\,[s^n]$
5. If φ is a formula of the form $\psi_1 \wedge \psi_2$, then:
   $i^{n+1} \vDash \varphi\,[s^n]$ if and only if $i^{n+1} \vDash \psi_1\,[s^n]$ and $i^{n+1} \vDash \psi_2\,[s^n]$
6. If φ is a formula of the form $\exists v^m\,\psi$, where $m \leq n$, then:
   $i^{n+1} \vDash \varphi[s^n]$ if and only if there is x^m in d^{m+1} such that $i^{n+1} \vDash \psi[s^n(v^m/x^m)]$
   where d^{m+1} is the domain of pluralities of level m encoded in d^{n+1} and $s^n(v^m/x^m)$ is a variant of s^n, namely an assignment just like s^n, with the possible exception that $s^n(v^m/x^m)$ assigns x^m to v^m.

Frege and Dedekind on recursive definitions

Frege (1879) and Dedekind (1888) discovered that recursive definitions can be turned into explicit ones by generalization over "collections" of the entities related by the recursive definition. Consider the case of addition in arithmetic. Using the prime symbol for the successor operation, $\textsc{Add}(x, y, z)$ ("z is a sum of x and y") can be defined recursively as follows:

(i) $\textsc{Add}(x, 0, x)$
(ii) $\textsc{Add}(x, y, z) \rightarrow \textsc{Add}(x, y + 1, z + 1)$

Then $\textsc{Add}(x, y, z)$ can be defined explicitly as follows:

$$\textsc{Add}(x, y, z) \leftrightarrow_{\text{def}} \forall R\big(\forall u R(u, 0, u) \wedge \forall u, v, w(R(u, v, w) \\ \rightarrow R(u, v', w')) \rightarrow R(x, y, z)\big)$$

Tarski (1935) realized that his own recursive definition of satisfaction could be turned into an explicit one in this way. The same obviously goes for his later definition of truth in a model. (See Appendix B.1 of Linnebo and Rayo 2012 for details.) This completes our proof sketch for the positive part of the Ascent Theorem.

As for the negative part, we already described how to prove this in the basic case, utilizing Plural Profusion. Assuming unrestricted comprehension for higher pluralities as well, it is easy to establish higher-level analogues of Plural Profusion, namely that there are more pluralities of level $n + 1$ than pluralities of level n. Equipped with this result, the observation we used to prove the basic case is easily extended to prove the arbitrary finite case.