2

Taking Plurals at Face Value

2.1 Some prominent views of plural sentences

Many natural languages contain a grammatical distinction between singular and plural expressions.* Consider these examples:

(2.1) John is hunting.

(2.2) The gnus are gathering.

When available, plural expressions can play a critical role in thought and language. On the one hand, by grasping their meaning and deploying them, we are able to think and speak about many as well as about one. For instance, we are able to sort objects into collections and communicate important information about such collections. On the other hand, plural expressions have logical properties that generate valid patterns of reasoning through which we organize and extend our knowledge about collections of objects, for example:

(2.3) (a) The gnus are gathering.
(b) The gnus are the animals being hunted.
(c) The animals being hunted are gathering.

These patterns of reasoning go beyond those studied and systematized in traditional first-order logic, forming the subject matter of a new branch of logic known as *plural logic*.

Following the lead of George Boolos's seminal work, research on plural logic has flourished in recent decades.[1] It has also begun to influence

* Sections 2.1–2.5 draw from Florio and Linnebo 2018.

[1] See, e.g., Boolos 1984b, Boolos 1985a, Yi 1999, Oliver and Smiley 2001, Rayo 2002, Linnebo 2003, Yi 2005 and Yi 2006, McKay 2006, and Oliver and Smiley 2016.

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linguistic semantics, where plurals have received considerable attention since the 1980s.[2]

Although this focus on plurals is a relatively recent phenomenon, semantic questions concerning plurals were already entertained by the founders of modern logic.[3] Gottlob Frege, for instance, addressed the question of the proper logical analysis of sentences with a plural subject, such as:

(2.4) Socrates and Plato are philosophers.

He writes:

> Here we have two thoughts: Socrates is a philosopher and Plato is a philosopher, which are only strung together linguistically for the sake of convenience. Logically, Socrates and Plato is not to be conceived as the subject of which being a philosopher is predicated.
>
> (Letter to Russell of 28 July 1902, in Frege 1980, 140)

In effect, Frege proposes to eliminate plurals and analyze (2.4) as:

(2.5) Socrates is a philosopher and Plato is a philosopher.

However, he realizes that this strategy isn't always available. Sentences such as (2.6) and (2.7) are not amenable to the conjunctive analysis proposed for (2.4).

(2.6) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.7) The Romans conquered Gaul.

Frege remarks:

> Here we must regard *Bunsen and Kirchhoff* as a whole. 'The Romans conquered Gaul' must be conceived in the same way. The Romans here are the Roman people, held together by custom, institutions, and laws.
>
> (Frege, ibidem)

[2] See Nicolas 2008 and Moltmann 2016 for applications of plural logic to linguistic semantics. For some research in linguistic semantics particularly relevant to our project, see Link 1983, Link 1998, Schein 1993, Schwarzschild 1996, Moltmann 1997, and Landman 2000.

[3] For historical details, see Oliver and Smiley 2016, Chapter 2.

Elsewhere he explains that, in (2.7), 'the Romans' must be regarded as a proper name whose logical function is to stand for an object (Frege 1980, 95).

While Frege understands "wholes" in broadly mereological terms—an approach to which we will return shortly—various alternatives, such as sets and groups, have been suggested in the subsequent literature. Let us briefly consider the appeal to sets.

The most famous advocate of this approach is Willard Van Orman Quine. One of the sentences he grapples with is known as the *Geach-Kaplan sentence*:[4]

(2.8) Some critics admire only one another.

According to Quine, by "invoking classes and membership, we can do justice to [the Geach-Kaplan sentence]" (Quine 1982, 293). He proposes what amounts to the following analysis, or, as he puts it, "regimentation":[5]

(2.9) There is a non-empty set such that every element of the set is a critic who admires someone and everyone she admires is an element of the set other than herself.

(2.10) $\exists s[\exists x\, x \in s \wedge \forall x(x \in s \rightarrow (x \text{ is a critic} \wedge \exists y\, x \text{ admires } y \wedge \forall y(x \text{ admires } y \rightarrow (y \in s \wedge x \neq y))))]$

Quine's use of set theory to eliminate plurals exposes him to an objection (see Boolos 1984b, 440). Consider the following sentence, which appears to be a set-theoretic truism:

(2.11) There are some sets such that any set is one of them if and only if that set is not an element of itself.

[4] As shown by Boolos, who credits David Kaplan, there is no correct paraphrase of this sentences comprising only singular vocabulary and the predicates occurring in it (Boolos 1984b, 432–3).

[5] We return in Sections 2.7 and 3.1 to a discussion of the important Quinean notion of regimentation, which differs from the familiar philosophical notion of analysis.

It is reasonable to demand that no proper regimentation of this sentence render it obviously false.[6] However, a strict application of Quine's set-theoretic paraphrase would turn (2.11) into (2.12), which is inconsistent:

(2.12) There is a non-empty set *x* such that, for every set *y*, *y* is an element of *x* if and only if *y* is a not an element of *y*.

Can Quine's strategy be salvaged by using a different paraphrase? Perhaps Quine is right that plural terms should be understood as "wholes" that are set-theoretic in character. But such "wholes" need not be sets; they can be collections of a more general sort. This provides a response to the objection presented above, since it licences this consistent regimentation of (2.11):

(2.13) There is a non-empty collection *c* such that, for every set *y*, *y* is a member of *c* if and only if *y* is not an element of itself.

However, this approach faces an immediate "revenge problem". How should we analyze the following variant of (2.11)?

(2.14) There are some collections such that any collection is one of them if and only if that collection is not a member of itself.

James Higginbotham aptly labels this style of objection *the paradox of plurality* (1998, 17). We provide a detailed discussion in Section 3.4.

In linguistics, an influential analysis of plurals is that of Godehard Link, who invokes mereological sums. Central to his analysis is a special mereological relation ($\leq$), corresponding to the notion of *individual parthood*. This notion is not to be confused with that of material parthood. For example, in the individual sense of the mereological vocabulary, Annie is an atomic part of the mereological sum of Annie and Bonnie. Here Annie is an atom, namely an individual with no other individual as part. In the material sense, by contrast, Annie is obviously not an atomic part of the sum of Annie and Bonnie, as she has proper material parts.

Link's proposal is to use mereology in this individual sense and analyze a plurality in terms of the mereological sum of its members. For example, the plurality of Annie and Bonnie would be analyzed in terms of the mereological sum of the two girls. In this setting, the relation of 'being one of' is

[6] According to the view defended in Linnebo 2010, a natural reading of (2.11) is false, but only for the *non-obvious* reason that every plurality must be extensionally definite, or properly circumscribed, which contrasts with the extensional indefiniteness of the notion of a self-identical set. This approach will be explored in Chapter 12.

best analyzed as 'being an atomic part of' ($\leq_{At}$). Let '+' stand for the binary operation of mereological sum in the individual sense. And let $\sigma x.\varphi(x)$ be the mereological sum, again in the individual sense, of the objects satisfying the formula $\varphi(x)$.[7] Then some of the plural sentences we have encountered may be analyzed as follows:[8]

(2.15) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.16) $F(b + k)$

(2.17) The Romans conquered Gaul.

(2.18) $C(\sigma x.R(x), g)$

(2.19) There are some sets such that any set is one of them if and only if that set is not an element of itself.

(2.20) $\exists x\,[\forall y(y \leq_{At} x \rightarrow \text{Set}(y)) \wedge \forall y(\text{Set}(y) \rightarrow (y \leq_{At} x \leftrightarrow y \notin y))]$

Finally, let us mention a singularist strategy based on a neo-Davidsonian analysis of predication in terms of events (broadly understood to include states).[9] This strategy eliminates a plural subject by reducing it either to a collection serving as agent of the underlying event or to the single co-agents of that event, where a co-agent is any object that participates in the event as a subject. Here is how the second version of the strategy may be applied to one of Frege's examples.

(2.15) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.21) There is an event *e* of laying the foundations of spectral analysis such that Bunsen is a co-agent of *e*, Kirchhoff is a co-agent of *e*, and there is no other co-agent of *e*.

Are any of these singularist analyses of plurals successful? This question is discussed in Chapter 3, which provides a detailed assessment of the prospects for singularism. Whether singularism is a viable option, we argue, depends on some hard theoretical questions concerning absolute generality and the correct plural logic. Now, we would like to consider an altogether different approach to plurals.

[7] If desired, the notion of sum can be defined in terms of the parthood relation by exploiting the fact that a sum is the minimal object whose parts include the things to be summed.

[8] For more details and applications of the mereological framework, see Link 1983, Link 1998, Moltmann 1997, Champollion and Krifka 2016, and Champollion 2017. We explore the relation between plurals and mereology in Chapter 5.

[9] See, e.g., Higginbotham and Schein 1989, and, for more recent implementations, Landman 2000 (especially Lecture Six) and Champollion 2017 (Chapter 2).

2.2 Taking plurals at face value

Boolos rejects all the singularist strategies, favoring instead an approach that takes plurals at face value. Thus he completely rejects Quine's attempt to analyze plural discourse in terms of sets. He writes:

> Abandon, if one ever had it, the idea that use of plural forms must always be understood to commit one to the existence of sets [. . .] of those things to which the corresponding singular forms apply.
>
> There are, of course, quite a lot of Cheerios in that bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? [. . .]
>
> It is haywire to think that when you have some Cheerios, you are eating a set [. . .]. [I]t doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all. (Boolos 1984b, 448–9)

In fact, Boolos's rejection of singularism has a distinguished pedigree featuring, most prominently, Russell (1903).[10] Russell distinguished between a *class as one* and a *class as many*. A class as one is a single object that may have a multiplicity of members. Objects of this kind are the subject matter of traditional first-order set or class theory. By contrast, a class as many is a multiplicity of objects *as such*: there need not be a single entity that represents, collects, or goes proxy for the objects that make up the multiplicity. Russell emphasized the usefulness of this second way of thinking about multiplicities. More recently, Max Black (1971) and Peter Simons (1982, 1997) have advocated a treatment of plurals in the spirit of classes as many.[11]

What is the broader significance of Boolos's attack on singularist analyses and of Russell's earlier pluralist approach based on the notion of classes as many? At the heart of their remarks is the simple idea that plurals should be taken at face value. That is, we should allow certain forms of plural discourse in our regimentation. Frege, Quine, and others were simply wrong to think that plurals should be paraphrased away. Rather, plurals deserve to be understood in their own terms by allowing the use of plural expressions in

[10] See Klement 2014 for a recent discussion of Russell's view.
[11] Again, see Oliver and Smiley 2016, Chapter 2, for more historical details.

our regimenting language. This is the key idea behind plural logic. We outline some basic aspects of plural logic in the next two sections. First, we introduce a formal language for plural logic. Then, we provide a basic deductive system that characterizes correct reasoning in this language. The semantics for this system is discussed in Part III of this book. Some of the most interesting and philosophically significant questions concerning plural logic arise, as we will see, in connection with its semantics.

Let us be clear about what is at stake in the debate about whether plurals should be taken at face value. It is one thing to observe that primitive plural resources are found in many natural languages and quite another to accept these expressive resources as legitimate, or even indispensable, for scientific purposes. While no one disputes the former, the latter is controversial, as the views of Frege and Quinean on plurals illustrate. Speaking for ourselves, we grant that there is a presumption in favor of taking resources available in natural language to be scientifically legitimate. But there may be exceptions to this general rule. Quine thought that metaphysical modality provided an example. Another possible example is linguistic tense, which appears to presuppose a standard of absolute simultaneity, in conflict with the special theory of relativity. How, then, can we bridge this gap between availability in natural language and scientific legitimacy or even indispensability? Some arguments purporting to bridge the gap will be discussed in Chapter 3 and in Part II. Although we will find many of these arguments to be weak, we will develop and defend one argument having to do with the role of plurals in the explanation of set theory.

2.3 The language of plural logic

We now describe a language that may be used to regiment a wide range of natural language uses of plurals. It captures Boolos's and Russell's suggestion and enables us to represent many valid patterns of reasoning that essentially involve plural expressions. This language is associated with what is known in the philosophical literature as PFO+, which is short for *plural first-order logic plus plural predicates*. In one variant or another, it is the most common regimenting language for plurals in philosophical logic.[12]

We start with the the standard language of first-order logic and expand it by making the following additions.

[12] We adopt the notation for variables used in Rayo 2002 and Linnebo 2003. An ancestor of this notation is found in Burgess and Rosen 1997. Other authors represent plural variables by

A. Plural terms, comprising plural variables (*vv*, *xx*, *yy*, ..., and variously indexed variants thereof) and plural constants (*aa*, *bb*, ..., and variants thereof), roughly corresponding to the natural language pronoun 'they' and to plural proper names, respectively.
B. Quantifiers that bind plural variables ($\forall vv$, $\exists xx, \ldots$).
C. A binary predicate $\prec$ for plural membership, corresponding to the natural language 'is one of' or 'is among'. This predicate is treated as logical.
D. Symbols for collective plural predicates with numerical superscripts representing the predicate's arity ($P^1, P^2, \ldots, Q^1, \ldots$, and variously indexed variants thereof). Examples of collective plural predicates are '... cooperate', '... gather', '... surround ...', '... outnumber ...'. For economy, we leave the arity unmarked.

Let $\mathcal{L}_{\text{PFO+}}$ be the language just introduced. The fragment of this language containing items A-C, that is, $\mathcal{L}_{\text{PFO+}}$ minus plural predicates, is the language of the subsystem of PFO+ known as PFO. The following chart summarizes which linguistic items are added to the standard language of first-order logic to obtain PFO+.

type of expression	natural language equivalent	symbolization
plural variables	$\text{they}_1, \text{they}_2, \ldots$	$vv, vv_0, \ldots, xx, \ldots$
plural constants	the Hebrides, the Channel Islands[13]	$aa, bb, \ldots, aa_1, \ldots$
plural quantifiers	there are some (things)	$\exists vv, \exists xx, \ldots$
	whenever there are some (things)	$\forall vv, \forall xx, \ldots$
plural membership	is one of, is among	$\prec$
collective plural predicates	cooperate, gather, surround, outnumber	$C(xx), G(vv), S(xx, y), O(xx, yy)$

The recursive clauses defining a well-formed formula are the obvious ones. However, some clarifications about the language are in order.

means of different typographical conventions: boldface letters (Oliver and Smiley), capitalized letters (McKay), or singular variables pluralized with an 's' (Yi).

[13] These purported examples of plural terms are controversial; for an argument that they are best treated as semantically singular, see Rumfitt 2005, 88. Additional examples can be found in Oliver and Smiley 2016, 78–80.

First, our language has two types of variable: singular and plural. It is also possible to use plural variables only and regard the singular as a limiting case of the plural. (See Section 5.3 for discussion.)

Second, one may require a rigid distinction between the types of argument place of predicates. An argument place that is open to a singular argument could be reserved exclusively for such arguments. A similar restriction could be imposed on argument places open to plural arguments. Would this rigid distinction between singular and plural argument places reflect a feature of natural language? Different natural language predicates suggest different answers. Some predicates are flexible, combining felicitously with both singular and plural terms. Examples include 'own a house', 'lifted a boat', or, as in Frege's example, 'laid the foundations of spectral analysis'. (Of course, the conjugations of the verbs will have to be adjusted.) Other predicates appear to lack this flexibility, combining felicitously *only* with plural terms, as in 'cooperate with one another' and 'are two in number'. There is an interesting linguistic question as to the source of these felicity judgments: are they of syntactic, semantic, or pragmatic origin? We don't wish to take a stand on these matters. For our purposes, we can leave this question open, noting that the two kinds of argument place—apparently flexible and apparently inflexible—suggest different regimentation strategies, namely to admit flexible plural predicates, or not.[14]

Third, collective plural predicates are contrasted with distributive ones, such as 'are students', 'visited Rome', 'are prime'. Roughly speaking, these are predicates that apply to some things if and only if they apply to each of those things. How best to make this precise will depend on one's stand on the issue of flexible plural predicates mentioned just above. A flexible plural predicate P is distributive just in case the following equivalence holds:

$$P(xx) \leftrightarrow \forall x(x \prec xx \rightarrow P(x))$$

A slight modification is needed for inflexible plural predicates. Let P^s be the singular analogue of P. Then an inflexible plural predicate P is distributive just in case the following equivalence holds:

[14] The possibility of flexible plural predicates raises deep and interesting questions. In the philosophical and logical tradition, it is widely assumed that if an expression can be replaced by another expression *salva congruitate* in *one* context, then it can be so replaced in *all* contexts. This assumption of "strict typing" is true of the language of first-order logic, as well as of standard presentations of second-order logic. However, the assumption fails if some, but not all, plural predicates are flexible.

$$P(xx) \leftrightarrow \forall x(x \prec xx \rightarrow P^s(x))$$

Finally, if a plural predicate has no singular analogue (as is arguably the case for 'cooperate with one another' and 'are two in number'), then it is collective by default.[15]

Owing to these definitions, distributive plural predicates can be obtained by paraphrase from their corresponding singular forms. Such predicates can therefore be omitted from PFO+ without loss of expressibility—although admittedly with some violence to style.

Other useful notions can be obtained by paraphrase. One is the many-many relation of plural inclusion, symbolized as '$\preccurlyeq$' and defined thus:

$$xx \preccurlyeq yy \leftrightarrow_{\text{def}} \forall z(z \prec xx \rightarrow z \prec yy)$$

This relation is expressed by 'are among', as used in 'Annie and Bonnie are among the students'. Then, according to the definition, Annie and Bonnie are among the students just in case anything that is one of Annie and Bonnie is one of the students. Another notion is plural identity (symbolized as '$\approx$'), which can be defined as mutual plural inclusion.[16] In symbols:

$$xx \approx yy \leftrightarrow_{\text{def}} (xx \preccurlyeq yy \wedge yy \preccurlyeq xx)$$

That is, two pluralities are identical just in case they are coextensive.

To illustrate the use of PFO+, let us provide some examples of regimentation.

(2.22) Some students cooperated.

(2.23) $\exists xx\,(\forall y(y \prec xx \rightarrow S(y)) \wedge C(xx))$

(2.24) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.25) $\exists xx\,(\forall y(y \prec xx \leftrightarrow (y = b \vee y = k)) \wedge L(xx))$

[15] What is the status of these equivalences? If PFO+ is to capture entailment relations in natural language, we must regard them as analytic (or near enough). This is because, for example, 'Annie and Bonnie visited Rome' entails 'Annie visited Rome'. Notice that our definition of distributivity takes the form of (analytic) equivalences. Some authors (e.g. McKay 2006, 6) tie distributivity solely to the left-to-right implication. For discussion and references, see Oliver and Smiley 2013, 114–15. For an overview of linguistic treatments of distributivity, see among others Winter and Scha 2015 and Champollion forthcoming.

[16] Of course, if flexible predicates are allowed, then plural identity can arguably be expressed by the ordinary identity predicate '='.

(2.26) Some critics admire only one another.

(2.27) $\exists xx\,(\forall x(x \prec xx \rightarrow C(x)) \wedge$
$\forall x \forall y[(x \prec xx \;\wedge\; A(x, y)) \rightarrow (y \prec xx \;\wedge\; x \neq y)])$

We now turn to the basic proof-theoretic aspects of plural logic.

2.4 The traditional theory of plural logic

The formal system PFO+ comes equipped with logical axioms and rules of inference aimed at capturing correct reasoning in the fragment of natural language that is being regimented. The axioms and rules associated with the logical vocabulary of ordinary first-order logic are the usual ones. For example, one could rely on introduction and elimination rules for each logical expression. The plural quantifiers are governed by axioms or rules analogous to those governing the first-order quantifiers.

Plural logic is often taken to include some further, very intuitive axioms. First, every plurality is non-empty:

(Non-empty) $\forall xx \exists y\; y \prec xx$

Then, there is an axiom scheme of indiscernibility stating that coextensive pluralities satisfy the same formulas:

(Indisc) $\forall xx \forall yy[xx \approx yy \;\rightarrow (\varphi(xx) \;\leftrightarrow\; \varphi(yy))]$

We need to make some remarks. First, the formula φ may contain parameters. So, strictly speaking, we have the universal closure of each instance of the displayed axiom scheme. Henceforth, we assume this reading for similar axiom schemes, including the one below, and for axioms with free variables in general. Second, as customary, we write $\varphi(xx)$ for the result of replacing all free occurrences of some designated plural variable vv with 'xx' whenever 'xx' is substitutable for vv in φ (see for example Enderton 2001, 113).) Third, (Indisc) is a plural analogue of Leibniz's law of the indiscernibility of identicals, and as such, the scheme needs to be restricted to formulas $\varphi(xx)$ that don't set up intensional contexts.

Finally, there is the unrestricted axiom scheme of *plural comprehension*, an intuitive principle that provides information about what pluralities there are. For any formula $\varphi(x)$ containing 'x' but not 'xx' free, we have an axiom stating that if $\varphi(x)$ is satisfied by at least one thing, then there are the things each of which satisfies $\varphi(x)$:

(P-Comp) $$\exists x\varphi(x) \rightarrow \exists xx\forall x(x \prec xx \leftrightarrow \varphi(x))$$

We refer to an axiomatization of plural logic based on the principles just described as *traditional plural logic*. This is to emphasize its prominence in the literature.

Traditional plural logic can, of course, be challenged. We will be particularly concerned with unrestricted plural comprehension. A challenge to this axiom scheme will be examined in Chapters 11 and 12. To talk about some things, we presumably need to circumscribe the things in question. Perhaps this circumscription isn't a trivial matter. That is, perhaps some conditions $\varphi(x)$ fail to circumscribe some things. For example, the trivial condition '$x = x$' might fail to do so because there is no properly circumscribed lot of "all objects whatsoever". We will eventually take this kind of challenge seriously and develop an alternative, and slightly weaker, "critical" plural logic. However, for the time being we will work with traditional plural logic, which includes the unrestricted plural comprehension scheme.

2.5 The philosophical significance of plural logic

The significance of plural logic is not only linguistic: it is not exhausted by its helpfulness in capturing natural language reasoning involving plural expressions. Plural logic is *philosophically* significant in that it has a claim to provide a suitable framework in which various philosophical projects can be successfully developed. This philosophical significance largely depends on two features that plural logic has been thought to possess: first, plural logic is in some sense "pure logic"; second, it provides greater expressive power than first-order logic. These two alleged features are at the core of the common picture of plural logic and explain why it has become an important component of the philosopher's toolkit. In this section, we flesh out this picture and describe how it sustains the main philosophical applications of plural logic.

Many aspects of this common picture of plural logic will be challenged throughout the book. Although this calls into question some popular applications of plural logic, we also develop some new applications; in particular, we show how plural logic can be used to shed light on set theory.

The first alleged feature of plural logic concerns its status as "pure logic". Surveying and assessing the debate about what counts as pure logic would take us too far afield. In the present context, we find it more fruitful to regard logicality, not as an all-or-nothing feature of a system, but as a cluster of conditions that are of independent philosophical interest. There are at least three such conditions that might underwrite the philosophical significance of plural logic: *topic-neutrality*, *formality*, and *epistemic primacy*. Let us discuss each in turn.

Topic-neutrality is based on a simple, intuitive idea: logical principles should be applicable to reasoning about any subject matter. By contrast, other principles are only applicable to particular domains. The laws of physics, for instance, concern the physical world and do not apply when reasoning about natural numbers or other abstract entities. Plural logic seems to satisfy this intuitive notion of topic-neutrality: the validity of the principles of plural logic does not appear confined to specific domains. As partial evidence for the topic-neutrality of plural logic, one may point out that, when available, pluralization as a morphological transformation does not depend in any systematic way on the kind of objects one speaks about. For example, both concrete and abstract nouns exhibit plural forms. The same goes for many other categorial distinctions.[17]

Another mark of logicality is formality. Logical principles are often thought to hold in virtue of their form, not their content. There are different ways of articulating the notion of formality, some of which are tightly connected to the notion of topic-neutrality just discussed (see MacFarlane 2000). We focus on two conditions that tend to be associated with formality. One is that formal principles are *ontologically innocent*: they do not commit us to the existence of any objects.[18] Another is that formal principles *do not discriminate between objects*: they cannot single out particular objects or classes thereof.

[17] On a closer look, we must distinguish between the weaker claim that *some* system of plural logic has topic-neutrality and the stronger claim that plural logic *as formulated above* has this neutrality. The latter may be challenged while retaining the former, as noted in footnote 6 and further explored in Chapter 12. We have in mind the view defended by Yablo (2006) and Linnebo (2010), according to which every plurality is extensionally definite, or circumscribed, in a way that the entire universe is not. This means that the plural comprehension scheme must be restricted when the domain of discourse is the entire universe (e.g. the formula '$x = x$' does not define a plurality).

[18] Of course, the choice of a non-free logic requires the existence of one object.

Is plural logic ontologically innocent? In particular, are plural quantifiers ontologically innocent? The usual answer to these questions is affirmative. Plural quantifiers do not incur ontological commitments beyond those incurred by the first-order quantifiers. Plural logic indeed originated as an ontologically innocent alternative to second-order logic. This view is sustained by a particular semantics for plural logic—due to Boolos (1985a) —which differs from the set-based semantics ordinarily employed for logics of first and second order. To see how, let us briefly sketch Boolos's semantics.

The key feature of this semantics is that it adopts plural resources in the metatheory and uses them to represent the semantic values of the plural terms of the object language. On this semantics—which many philosophers now regard as the canonical one—the difference between singular and plural terms is explained, not on the basis of *what* these terms signify, but on the basis of *how* they signify. A plural variable is not interpreted as a set (or set-like entity) of objects in the first-order domain. Instead, it is interpreted directly as *many objects* in this domain, without the mediation of a set (or set-like entity). In other words, plural variables do not range over a *special domain* but range in a *special, plural way* over the usual, first-order domain. Since the range of plural variables is the first-order domain, the truth of sentences involving plural quantifiers does not seem to make ontological demands that exceed those made by sentences involving first-order quantifiers. In this sense, plural logic is said to be ontologically innocent.[19]

As noted above, there is another condition associated with formality: formal principles must not discriminate between objects. The standard way of making this condition precise is to claim that logical principles are those that remain true no matter how the non-logical expressions of the language are reinterpreted. This presupposes a distinction between logical and non-logical expressions of the language, which is typically captured by defining logical notions in terms of isomorphism invariance and then characterizing as logical the expressions that are suitably related to logical notions.[20] Alfred Tarski (1986) observed that isomorphism invariance captures the standard

[19] Boolos's semantics has been widely used in philosophical logic. See, among others, Yi 1999, Yi 2002, Yi 2005, and Yi 2006; Hossack 2000; Oliver and Smiley 2001 and Oliver and Smiley 2016; Rayo 2002; McKay 2006. Authors who use this semantics tend to emphasize the ontological innocence of the resulting logic.

[20] See Tarski 1986, Sher 1991, and McGee 1996. Denoting a logical notion has been claimed to be necessary but not sufficient for an expression to be logical. An additional semantic connection would be required (as argued, for instance, by McCarthy 1981 and McGee 1996; but see also Sagi 2015 for a critical evaluation of these arguments).

logical notions expressible in higher-order logic.[21] Thus higher-order logic should count as formal according to this way of explicating the notion of formality. It is natural to think that analogous arguments ought to apply to plural logic, delivering the result that plural quantification and plural membership are logical notions, and that plural logic too should count as formal.

The final mark of logicality, we recall, is epistemic. The thought is that logical notions and principles permit a special kind of *epistemic primacy*.[22] Logical notions can be grasped without relying on non-logical notions. Likewise, logical truths, if knowable, can be known independently of non-logical truths. Do the principles of plural logic enjoy this kind of epistemic primacy? Since some of these principles are counterparts of principles of first-order logic (for example, the introduction and elimination rules for the quantifiers), it is plausible to assume that they enjoy the same epistemic status as their first-order counterparts. However, plural logic encompasses distinctive principles—chiefly plural comprehension—and the question is whether *they* are subject to epistemic primacy. For the moment, let us simply record the fact that many philosophers find plural comprehension to be obviously true. For example, Boolos writes that every instance of comprehension "expresses a *logical* truth if any sentence of English does" (Boolos 1985b, 342). Similarly, Keith Hossack finds plural comprehension to be a "harmless *a priori* truth" and, together with the other axioms of plural logic, regards it as a genuine logical truth (Hossack 2000, 422).

If logicality is the first key feature of the common picture of plural logic, the second is expressive power. Because of its metalogical properties, first-order logic has well-known expressive limitations. In particular, important mathematical theories formulated in first-order terms are subject to non-standard interpretations. For example, first-order arithmetic has uncountable models, while first-order analysis and set theory have countable ones. So first-order logic badly fails to express the intended models of such theories. By contrast, plural logic is usually ascribed metalogical properties that lead to greater expressive power. Indeed, it is often held that, when formulated with the help of plural quantification, arithmetic, analysis, and set theory avoid the non-standard interpretations just mentioned.[23] The resulting view, which we dispute in Chapter 8, is that plural logic does better than first-order logic in securing a gain in expressive power.

[21] See also Lindenbaum and Tarski 1935.

[22] See, for example, how Frege frames his logicist project in Frege 1879, Frege 1884, and Frege 1893/1903.

[23] See footnote 3 on p. 152 for references.

To sum up: on the common picture, plural logic has two key features, logicality and expressive power. As noted above, instead of thinking of logicality as an all-or-nothing matter, we find it more fruitful to regard it as a cluster of conditions. We isolated three such conditions: topic-neutrality, formality, and epistemic primacy. Under formality, we distinguished two further conditions: formal principles are ontologically innocent, and they cannot single out particular objects or classes of objects.

2.6 Applications of plural logic

The philosophical significance of plural logic lies in its promise to provide an essential tool for various philosophical projects. An obvious such project is to provide an account of plurals in thought and language. There are less obvious uses of plural logic as well. We now wish to describe some particularly important applications to the philosophy of mathematics, metaphysics, and semantics. As will become clear, these applications rely on various aspects of the common picture of plural logic discussed above. Some of these aspects will be challenged in the course of the book, especially the ontological innocence and expressive power of plural logic (see Chapter 8) and its epistemic primacy (see Chapter 12).

There is a well-known technical result that sheds lights on many of these applications. As first shown by Boolos (1984b), monadic second-order logic can be interpreted in PFO. (Monadic second-order logic is the fragment of second-order logic that allows quantification into predicate position only when the predicate is monadic.) The converse is true as well: PFO can be interpreted in monadic second-order logic.[24] From a syntactic point of view, the two theories are therefore equivalent. However, this mutual interpretability by no means guarantees that the two systems share certain philosophically important features and, hence, that they are equivalent in their potential for philosophical applications. Since second-order logic has faced a number of criticisms that are usually thought to be avoided by plural logic, one might hope to be able to replace at least some uses of monadic second-order logic with corresponding uses of plural logic.[25]

[24] See Chapter 6 for a detailed discussion of the result.

[25] Second-order logic has been criticized on various grounds, e.g. for involving an illegitimate form of quantification, for being ontological committal, and for being too entangled with mathematics to count as pure logic (see Linnebo 2011 for a survey of the standard objections to second-order logic).

As developed in the work of Frege and his followers, logicism is the thesis that a significant portion of mathematics is analytic in the sense of being derivable from general logical laws and definitions. Second-order logic provides the standard framework for the development of logicism. Thus the success of Fregean logicism depends crucially on the logicality of second-order logic. Which features of logicality matter here? When discussing the philosophical significance of logicism, Frege and more recent logicists have tended to emphasize topic-neutrality, ontological innocence, and epistemic primacy. Since these are features that plural logic is alleged to have, this logic promises be of immediate relevance to the logicist project. One concern, which we discuss in Section 12.5, is that plural logic doesn't actually enjoy epistemic primacy but on the contrary carries non-trivial set-theoretic content.

There is a more clear-cut worry, however. The resources needed for the standard implementation of the logicist project exceed those of monadic second-order logic, and therefore those of plural logic. For instance, the statement of one of the main stepping stones of logicism, Hume's Principle, requires quantification over dyadic relations. The principle asserts that, for any two monadic second-order entities F and G, the number associated with F is identical with the number associated with G if and only if there is dyadic relation witnessing the equinumerosity of F and G.

Plural logic may still have an important role to play in logicism. First, there are alternative implementations of logicism that rely on plural logic coupled with a thin understanding of relations (see Boccuni 2013). Second, one might be able to capture quantification over relations by supplementing monadic second-order logic with ordered pairs obtained by first-order abstraction principles (see Shapiro and Weir 2000, Tennant 2007), by embracing extensions of plural logic like the one devised in Hewitt 2012a, or by regarding equinumerosity (or some kindred notion) as primitive (Antonelli 2010). Moreover, even if plural logic cannot sustain the full logicist project, it could still serve a more modest form of logicism, such as Boolos's *sublogicism*. As Boolos describes it, sublogicism is "the claim that there are (many) interesting examples of mathematical truths that can be reduced (in the appropriate sense) to logic" (Boolos 1985b, 332). His case for sublogicism relies on plural logic. It essentially involves a plural interpretation of Frege's definition of the ancestral of a relation (see Boolos 1985b).

Another important philosophical application of plural logic, underpinned by its alleged ontological innocence, concerns various eliminative projects in metaphysics. For example, plural logic has been used to eliminate reference to certain kinds of complex objects.[26] Instead of quantifying over tables, say, one may quantify plurally over mereological atoms of some appropriate kind, namely those "arranged tablewise". A sentence involving tables, such as 'some table is in the room', can thus be rendered as a sentence involving pluralities of mereological atoms, namely 'some mereological atoms arranged tablewise are in the room'. Since the predicate 'arranged tablewise' is a collective predicate, this eliminative strategy can be carried out in PFO+. An interesting question raised by this strategy is how *plural* quantification over complex objects should be treated (Uzquiano 2004b). Since we have already "used up" ordinary plural quantification to paraphrase singular talk of complex objects, eliminating plural talk of such objects requires additional resources. We would need a form of quantification that stands to plural quantification as plural quantification stands to singular quantification. The availability of such expressive resources is discussed in Chapter 9.

Relatedly, plural logic has been used to eliminate reference to abstract objects. In particular, quantification over sets can sometimes be replaced by plural quantification over concrete objects. This nominalist strategy is of interest also to non-nominalists. In set theory, for example, quantification over proper classes might be eliminated in favor of plural quantification over sets (see Uzquiano 2003 and Burgess 2004).

The next application of plural logic we would like to highlight has to do with semantics. An example was already mentioned in the previous section. While discussing ontological innocence, we outlined the semantics for plural logic developed by Boolos. The key idea was to employ plural resources in the metalanguage and interpret each plural variable as standing for one or more objects rather than a set or some set-like entity. Boolos's semantic insight is applicable in other contexts as well. Plural resources can also be used to formulate a semantics for first- and second-order logic (see Rayo and Uzquiano 1999, Rayo and Williamson 2003). As we discuss at length in Chapter 11, an important aspect of this semantics is that it enables us to capture interpretations of the language whose domain of quantification encompasses absolutely everything there is. Let us briefly explain.

In the usual set-theoretic semantics, domains of quantification are represented by sets. However, since there is no universal set in standard set theory, there is no way of representing interpretations whose domain encompasses absolutely everything. Arguably, this is problematic. For certainly it *seems* that absolutely general quantification is possible; consider, for example:

[26] See van Inwagen 1990, Hossack 2000, Rosen and Dorr 2002, and Uzquiano 2004b.

(2.28) The empty set has no elements.

(2.29) Everything is physical.

These are *prima facie* cases in which the quantifiers 'no' and 'every' range over absolutely everything there is.

If we accept that quantification over absolutely everything is possible, set-theoretic semantics appears inadequate in this respect. This apparent inadequacy might be overcome by developing the semantics with the help of plural logic. Rather than describing a domain as a set-like entity whose members constitute the range of quantification, one may describe it as *some objects*, without assuming that there is a single entity to which the elements of the domain all belong as members. How does this help us with absolute generality? Once we let a domain be a plurality of objects, it seems, we can capture absolute generality by means of the universal plurality, that is, the plurality of absolutely everything there is. The existence of a universal plurality is guaranteed by the plural comprehension scheme available in traditional plural logic, for example, by using the formula '$x = x$'.

It is noteworthy that the ability to do justice to absolute generality depends on the ontological innocence of plural logic, at least in the narrow sense that it introduces no new commitments to sets or other set-like objects. If plural logic was committed in this sense, our use of it to capture absolute generality would likely be undermined. For in that case, there could be no universal plurality, contrary to traditional plural logic. This can be shown under minimal assumptions by an argument analogous to that of Russell's paradox. Suppose that plural talk is not ontologically innocent, in the sense that the existence of a plurality xx requires the existence of a corresponding set (or set-like object) $s(xx)$. The correspondence between xx and $s(xx)$ is understood in terms of coextensionality: anything is one of xx if and only if it is a member of $s(xx)$. An assumption we need is that the membership relation for these sets (or set-like objects) is subject to a principle of separation.[27] Now let uu and $s(uu)$ be, respectively, a universal plurality and its corresponding set (or set-like object). By separation, there is an object r whose members are all and only the things in $s(uu)$ that are not members of themselves:

[27] In this context, we can state the principle as follows. Given any set (or set-like object) s and any condition $\varphi(x)$, there is a set (or set-like object) r whose members are all and only the members of s which satisfy the condition. That is, there is a set (or set-like object) r such that, for any x, x is a member of r if and only if x is a member of s and $\varphi(x)$.

$$\forall x(x \text{ is a member of } r \leftrightarrow (x \text{ is a member of } s(uu) \wedge x \text{ is not a member of } x))$$

Since *uu* is universal, so is *s*(*uu*). Thus anything is a member of *s*(*uu*) and, therefore, the members of *r* are all and only the things that are not members of themselves:

$$\forall x(x \text{ is a member of } r \leftrightarrow x \text{ is not a member of } x)$$

By instantiating the universal quantifier with *r*, we reach a Russell-style inconsistency:

$$r \text{ is a member of } r \leftrightarrow r \text{ is not a member of } r$$

The conclusion is that, if plural talk is not ontologically innocent in the mentioned sense, there cannot be a universal plurality. If there were such a plurality, there would be a corresponding set (or set-like object), leading to a version of Russell's paradox. But if there is no universal plurality, plural logic cannot serve to capture interpretations whose domain contains absolutely everything. Thus, an important application of plural logic in semantics would have to be renounced.

The last application we consider pertains to the philosophy of mathematics and relates to the expressive limitations of first-order languages mentioned in Section 2.5. Many attempts to overcome these expressive limitations resort to higher-order languages. In particular, second-order resources are often employed with an aim to provide a categorical axiomatization of the natural number structure, the real number structure, and certain initial segments of the hierarchy of sets. The ability to provide characterizations of this sort plays a major role in some philosophical accounts of mathematics, such as various forms of structuralism (see, for example, Hellman 1989 and Shapiro 1991).

However, the view that second-order logic is more expressive than first-order logic is not uncontroversial. It depends essentially on a particular semantics for second-order logic, which may be rejected. Since it is commonly assumed that plural logic does better than first-order logic in securing a gain in expressive power, plural logic has emerged as an appealing alternative to second-order logic in philosophy of mathematics. We critically assess this application of plural logic in Chapter 8.

2.7 Our methodology

This book is first and foremost a contribution to philosophical logic, although we occasionally aim to contribute to the philosophy of language and linguistics as well. Our primary interest is in exploring possible language forms and their philosophical significance. Indeed, what matters for philosophical purposes is often just *the availability* of certain language forms, rather than their *actual realization* in some natural language or other. For instance, the study of modalities exhibits a wide range of operators governing modal scope, some of which can be of philosophical use despite not having a correlate in natural language. Higher-order logic has also found a number of important applications even though it is controversial whether there is a genuine form of quantification into predicate position in natural language. Moreover, formal languages include predicates with arbitrarily high arity that do not correspond to any predicate of natural language. (There are presumably no primitive 17-adic predicates in English.)

This approach allows us to separate the study of possible language forms from the more straightforwardly empirical question of which of these language forms are in fact realized in natural language. We sometimes take a stand on the latter question. But there are also occasions when we set aside considerations of faithfulness to natural language in favor of an exploration of possible language forms, which is subject to fewer empirical constraints.

Even when we do consider natural language, our main focus is on regimentation rather than on a perfect representation of some underlying logical form, as this notion is understood in early analytic philosophy or in contemporary linguistic semantics.[28] The notion of regimentation we employ is Quinean in spirit.[29] To regiment a language is to paraphrase it into a fragment of ordinary or semi-ordinary language so as to lay bare structural features of relevance to the theoretical goals at hand, usually to address questions about logical consequence (see Chapter 3 for details). For example, our target in regimenting plural discourse into PFO or PFO+ is not to provide a faithful representation of *the* logical form underlying the discourse. The target is more modest: we aim to provide a representation of plural discourse that captures the logical features that are important in

[28] See, for instance, Pietroski 2016.

[29] See e.g. Quine 1960, 159.

the given context of investigation. This means that a regimentation might not capture all the logical relations observed in the regimented language. However, we do make the converse demand that all the logical relations obtaining in the regimenting language be reflected in the regimented one.[30] In any case, departures from what might be regarded as *the* logical form of a sentence will be justified on the basis of particular theoretical interests.

Our emphasis on the availability of possible language forms rather than their realization in natural language contributes to our goal of putting some philosophically interesting questions into sharper focus. In fact, the richness of natural language might be a hindrance to this goal. Consider the case of logical paradoxes. The significance of these paradoxes will of course depend on one's project. Paradoxes may not be of paramount importance in linguistic investigations concerned with a faithful representation of a given language, where the paradoxical aspects may simply be regarded as traits of that language. But paradoxes are clearly relevant to a project in philosophical logic that aims to explore possible language forms and identify those that can serve certain purposes in scientific theorizing. In that context, modeling a paradox is not enough: the paradox must somehow be resolved.

Our concern with paradoxes also explains the emphasis we place on absolute generality. Many of the arguments leading to paradox depend essentially on the assumption that quantification over absolutely everything is possible and that an adequate model theory must do justice to this possibility. The role played by absolute generality in the semantics of plurals is clarified in Chapter 7, where we discuss the connection between such generality and different approaches to model theory.

Additional questions that will benefit from an investigation of the kind we pursue concern ontological commitment, the legitimacy of various forms of quantification, the determinacy of plural quantifiers, and the relation between plurals and modalities. These are some of the main themes of this book.

[30] We will refer to this requirement as *logical adequacy* (see Section 3.1).