# 5
# Plurals and Mereology

In the previous chapters, we discussed two ways of conveying information about many objects simultaneously. The first uses primitive plurals, while the second uses sets. We now examine a third alternative based on mereology.

Mereology is the theory of part-whole relations. Instances of such relations are easy to find. Consider a hydrogen atom that is part of a water molecule, which in turn is part of the contents of a bottle. Mereology aims to capture the general principles governing various relations of parthood. For instance, we may ask whether it follows that the hydrogen atom, in the mentioned example, is part of the contents of the bottle. The intuitive answer is affirmative. This points to a general principle that is assumed to hold of most relations of parthood, namely transitivity.

In this chapter, we present a basic development of mereology and compare it with plural logic. As we will see, the formal relation between these two systems is analogous to that between plural logic and the simple set theory of Section 4.1. This raises questions parallel to those encountered in the preceding chapter. Can we eliminate plurals in favor of mereology? Can we eliminate mereology in favor of plural logic? Or are there reasons to retain both systems?

## 5.1 Mereology

Let us begin by developing a basic formal framework for mereology. We start with the usual language of first-order logic and expand it with a new primitive predicate '$\leq$' for parthood. So we read '$x \leq y$' as '$x$ is part of (or equal to) $y$'. The resulting language is one-sorted: its only variables are the ordinary first-order ones.

The new primitive predicate allows us to define a number of important mereological relations. First, there is proper parthood, which we write

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as '$<$' and define by letting '$x < y$' abbreviate '$x \leq y \wedge y \neq x$'.[1] For example, England is a proper part of the United Kingdom. Next, let us say that $x$ and $y$ *overlap* when they have a common part: $\exists z(z \leq x \wedge z \leq y)$; we symbolize this as '$x \circ y$'. For example, Scandinavia and the European Union overlap, as both have Denmark as a part. Finally, let us say that $x$ and $y$ are *disjoint* when they do not overlap; we symbolize this as '$x \perp y$'. For example, the United Kingdom and Scandinavia are disjoint.

We turn now to the theory of mereology. For our purposes, the most relevant theory is so-called *Classical Extensional Mereology* (sometimes also known as *General Extensional Mereology*). We first adopt axioms stating that $\leq$ is a partial order (that is, $\leq$ is reflexive, transitive, and anti-symmetric).[2] Next, we adopt the axiom of Strong Supplementation, which states that when $x$ is not part of $y$, there is a part $z$ of $x$ that does not overlap $y$:

$$x \nleq y \rightarrow \exists z(z \leq x \wedge z \perp y)$$

For example, since the European Union is not part of Denmark, the former must have a part that is disjoint from the latter. Finally, we adopt an axiom scheme asserting the existence of arbitrary mereological sums (or "fusions", as they are also called). Suppose some object is $\varphi$. Then there is an object $y$ that overlaps something $z$ just in case $z$ overlaps an object that is $\varphi$; $y$ is said to be *the sum of all objects that are* $\varphi$. The existence of arbitrary sums is captured by the following axiom:

(M-Sum) $$\exists x\varphi(x) \rightarrow \exists y\forall z\big(y \circ z \leftrightarrow \exists w(z \circ w \wedge \varphi(w))\big)$$

We often denote sums by means of the familiar '+' symbol; for example, the sum of $a$ and $b$ is written '$a + b$'.[3]

## 5.2 Can mereology represent the plural?

It is fairly obvious why set theory is an attractive tool for conveying information about many objects simultaneously. Instead of talking about some objects, we can talk about the set whose elements are precisely these objects.

[1] Given antisymmetry (see footnote 2), an equivalent reading of '$x < y$' is '$x \leq y \wedge y \nleq x$'.
[2] Recall that a relation $R$ is said to be anti-symmetric if and only if $\forall x\forall y(Rxy \wedge Ryx \rightarrow x = y)$.
[3] Formally, $y$ is said to be the sum of $a$ and $b$ if and only if $\forall z(y \circ z \leftrightarrow z \circ a \vee z \circ b)$.

Given this set, we can always retrieve the objects in question, namely as the elements of the set. Above, we described mereology as an alternative to set theory (and primitive plurals) for the purpose of conveying information about many objects simultaneously. It is less obvious how mereology can serve this purpose. When we consider the mereological sum of some objects, we *cannot* in general retrieve the objects with which we started.

For an example of this phenomenon, consider the following two pluralities: Russell and Whitehead, and the molecules of Russell and Whitehead. These are obviously entirely different pluralities. While the former things are two in number, the latter things are far more numerous. Yet the two pluralities appear to have one and the same mereological sum. Indeed, to overlap Russell or Whitehead comes to the same thing as overlapping one of the molecules of these two logicians.

This suggests that the mereological sum of some objects is insufficient to represent these objects. An example by Oliver and Smiley (2001) makes the problem vivid. Consider the following inference:

$$\frac{\text{Russell and Whitehead were logicians}}{\text{The molecules of Russell and Whitehead were logicians}} \tag{5.1}$$

Because of the distributive predicate 'were logicians', the conclusion is false and hence the argument is invalid. Suppose we want to represent some objects by means of their mereological sum. This representation of the argument seems to yield a different logical verdict. As Oliver and Smiley remark:

> 'Whitehead and Russell' and 'the molecules of Whitehead and Russell' represent different decompositions of the same sum, but giving them that sum as their common reference forces the conclusion that the molecules of Whitehead and Russell were logicians. (2001, 293)

So they conclude that "mereological sums or fusions are ineligible" for the task of representing many objects simultaneously. Similar examples have been put forward by others (see, for example, Rayo 2002, 444–5, and McKay 2006, 42).

In light of these considerations, it may be surprising that mereology is a far more popular tool among linguists interested in plurals than set theory. Suppose we start with some objects. Whatever the merits of set-theoretic representations in general, the set of these objects at least enables us to retrieve the objects in question. By contrast, there appears to be no guarantee that we can retrieve the objects with which we began from their mereological sum. As we just saw, one and the same sum can be obtained by taking the sums of two logicians and of their many molecules.

However, this problem isn't fatal for the project of using mereology to represent many objects simultaneously. To see why, consider three *atomic* particles, say $a$, $b$, and $c$ (where by "atomic" we mean that they have no proper parts). To talk about the three atoms simultaneously, we may talk about their sum $a + b + c$. Given this sum, we can retrieve the three particles that jointly compose it: there is a unique way to break this sum down into its three atomic parts.[4] By talking about this single sum, we can therefore convey information about its three atomic parts. For example, the information that the three particles are collinear can be conveyed by saying that there is a line on which each atomic part of the sum lies.

Thus, provided that each of the objects in question is a mereological atom, mereology is a perfectly good tool for talking about all these objects simultaneously. But what if we wish to talk simultaneously about many objects that are not atomic but have proper parts? If we could somehow regard each object as an atom, the use of mereology to represent pluralities would be available more generally. Might this be possible?

A solution, developed and defended by Link (1983, 1998), goes as follows. Even if the objects with which we start have material parts, they can figure as atoms in a different sense: each is, in Link's phrase, an *individual atom*. That is, each object is an atom with respect to a different relation of parthood, namely *individual parthood*. While many complaints against mereological representations of plurals are appropriate for the ordinary notion of parthood, they do not apply to Link's notion.

Let us look at an example. In the material sense, the sum of Russell and Whitehead is the same as the sum of the molecules of Russell and Whitehead. This is not true, however, if the mereological notions are construed according to the relation of individual parthood. In that sense, the sum of Russell and Whitehead is the sum of Russell and Whitehead *conceived as atomic individuals*, that is, taken as atoms in the domain of quantification. This means that, in the individual sense, Russell and Whitehead are the only proper parts of the sum of Russell and Whitehead. It follows that, in the individual sense, the sum of Russell and Whitehead is *not* identical with

[4] Readers may find it an interesting exercise to prove this from the axioms of Classical Extensional Mereology.

the sum of their molecules. The former but not the latter has Russell and Whitehead as its only proper parts.

More generally, individual mereology starts with a domain of individuals that are treated as mereological atoms, ignoring other mereological relations in which those individuals may stand. A mereological structure is then defined on top of that domain. The relation of individual parthood satisfies the axioms of Classical Extensional Mereology. In addition, it satisfies the principle of Atomicity, which states that everything has an atom among its parts. Formally:

(M-Atomicity) $$\forall x \exists y(\mathrm{At}(y) \wedge y \leq x)$$

where '$\mathrm{At}(y)$' abbreviates '$\neg\exists z\, z < y$'. We call the resulting theory *Atomistic Classical Extensional Mereology*.

In fact, mereological sums have some advantages vis-à-vis sets, which have motivated their use in semantics. First, mereological sums are presumably just as concrete as their parts. While the set of Russell and Whitehead is frequently taken to be abstract, the sum of Russell and Whitehead is plausibly taken to be concrete. So, if we want our semantics to assign concrete entities to certain ordinary expressions, this recommends using sums rather than sets for that purpose.

Second, we might want to assign the same semantic value to 'Alice' and 'the objects that are identical with Alice'. Mereology allows this, since the sum of a single object is identical to this very object. By contrast, standard set theory does not allow this kind of identification, since a singleton set is distinct from its sole element. In fact, this problem has occasionally motivated the adoption of a non-standard set theory that allows exactly this kind of identification (Schwarzschild 1996, 1).

The appeal to individual mereology does raise an obvious question, however. Is it *permissible* to invoke mereological notions in the individual sense? The question can be split into two. First, is it even logically coherent to speak in this way? Second, assuming that it is coherent, is this just a manner of speaking or do the described mereological sums really exist?

We defend the claims of logical coherence and existence in Sections 5.3 and 5.8, respectively. Suppose we are right. Then we can assume that each of our initial objects is an individual atom. So we may consider sums of individual atoms. This ensures that pluralities of these initial objects are uniquely represented by the corresponding individual sum.

## 5.3 One-sorted plural logic

There is no concern about the logical coherence of individual mereology. As we will now show, plural logic can be developed in a way that realizes precisely this structure.

Let us explain. As presented in Chapter 1, plural logic is based on a two-sorted language, since it contains two sets of variables. A singular variable ($x$, $y$, ...) ranges over a single object, while a plural variable ($xx$, $yy$, ...) ranges over one or more objects. It is also possible to dispense with the singular variables and provide a *one-sorted* version of plural logic. Our recent foray into mereology makes this straightforward: a one-sorted plural logic can be obtained as a mere notational variant of mereology. Instead of the usual singular variables, we use plural variables. And instead of the parthood predicate $\leq$, we use the symbol '$\preccurlyeq$' as a new primitive, though we continue to read '$xx \preccurlyeq yy$' as "$xx$ are among $yy$". Finally, there is an identity predicate that takes plural arguments. Indeed, in the one-sorted plural language, this is the *only* identity predicate.

What is it for some objects $xx$ to comprise a single object (and in this sense be an individual)? We define '$Ixx$' as '$\forall yy(yy \preccurlyeq xx \rightarrow xx \preccurlyeq yy)$'. That is, $xx$ comprise just a single object if and only if $xx$ are contained in each of its "subpluralities", which means that $xx$ has no strictly smaller "subplurality".

How should we axiomatize plural logic in this one-sorted presentation? A straightforward but clumsy option is simply to translate the axiomatization already adopted (see Section 2.4) into the new one-sorted language. For some axioms, the result is not bad. For example, the axiom stating that every plurality is non-empty, $\forall xx \exists y\, y \prec xx$, translates as:

$$\forall xx \exists yy(Iyy \wedge yy \preccurlyeq xx)$$

But the translations of other axioms are needlessly long and unintuitive.[5]

A more elegant option is to exploit the close connection we have observed between one-sorted plural logic and the atomistic version of Classical Extensional Mereology. So let us simply adapt the axioms of the latter to the former. First, we lay down that $\preccurlyeq$ is a partial order and obeys the Strong

[5] Plural comprehension provides a good illustration. This axiom scheme translates as

$$\exists xx(Ixx \wedge \varphi(xx)) \rightarrow \exists xx(\forall yy(Iyy \rightarrow (yy \preccurlyeq xx \leftrightarrow \varphi(yy))))$$

This does not enjoy the immediate plausibility of its two-sorted analogue, namely (P-Comp).

Supplementation principle. Next, we require an analogue of atomicity, that is, that every plurality has a subplurality comprising just a single individual:

(P-Atomicity) $\forall xx \exists yy(Iyy \wedge yy \preccurlyeq xx)$

Finally, we require the existence of arbitrary sums. One way to implement this requirement is by adopting a principle to the effect that, for every instantiated condition $\varphi(xx)$, there is a unique smallest plurality $zz$ that includes everything that satisfies the condition. That is, if $\varphi(xx)$ is instantiated, there are $zz$ such that:

(i) $zz$ include every $xx$ that satisfy the condition $\varphi(xx)$:

$$\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq zz)$$

(ii) $zz$ is the smallest plurality verifying requirement (i):

$$\forall ww(\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq ww) \rightarrow zz \preccurlyeq ww)$$

This principle can be given a more compact formalization as follows:

(P-Sum) $\exists xx\varphi(xx) \rightarrow \exists zz \forall ww\big(\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq ww) \leftrightarrow zz \preccurlyeq ww\big)$

An alternative way to require the existence of arbitrary sums is by adopting a plural analogue of (M-Sum). As we prove in Appendix 5.B, the two alternatives are in fact equivalent, given background assumptions that are currently in place.

The possibility of a one-sorted approach to plural logic is theoretically important. This approach is just as serious as its more familiar two-sorted cousin about the fact that plural terms can stand for many objects simultaneously. But this insight is represented in two very different ways. On the one-sorted approach, the insight is captured by means of the 'among'-predicate '$\preccurlyeq$'. Its argument places belong to the same sort, a sort that is given a plural interpretation. By contrast, on the two-sorted approach, the insight also has a syntactic manifestation in the sortal distinction between terms representing individual objects and terms representing many objects simultaneously. But clearly, this syntactic manifestation of the distinction between one and many is not obligatory. As we show shortly, many linguists prefer to do without it.

It should be unsurprising, in light of our discussion, that we can translate between the languages of one- and two-sorted plural logic. One direction is straightforward. Apart from the special logical predicate '$\preccurlyeq$', the one-sorted language is a sublanguage of the two-sorted one. So the translation from the one-sorted to the two sorted-language leaves unchanged every formula not containing '$\preccurlyeq$'. It remains to specify how this predicate can be translated. A plural 'among'-statement, '$xx \preccurlyeq yy$', can be translated as a generalized individual 'among'-statement, namely

$$\forall z(z \prec xx \rightarrow z \prec yy)$$

So we have the translation in one direction.

Let us now describe a translation $\tau$ in the reverse direction. Consider '$x \prec yy$'. This individual 'among'-statement can be translated as a plural 'among'-statement conjoined with a statement to the effect that some things comprise just a single object:

$$xx \preccurlyeq yy \wedge I(xx)$$

Quantification over individual objects can be translated as plural quantification restricted to singleton pluralities. Thus, '$\exists x\, \varphi$' is translated by $\tau$ as:[6]

$$\exists xx(Ixx \;\wedge\; \tau(\varphi))$$

Again, it turns out that both translations provide an interpretation of one theory in the other. Formal definitions and proofs are given in the appendices.

It is also no surprise that we can translate between one-sorted plural logic and Atomistic Classical Extensional Mereology. After all, we have formulated the former theory by simply adapting the axioms of the latter. It follows that all of the three theories considered in this section—the mereological one and plural logic with one or two sorts—are mutually interpretable. Again, see the appendices for details.

## 5.4 Classifying some ways to talk about the many

This chapter and the previous one have described three different ways to talk about the many. In addition to the use of primitive plural resources, we can use sets or (individual) mereology. We have seen that there are close connections between these different systems. But let us be more systematic.

[6] We assume a convention is in place to ensure that $xx$ does not occur in $\tau(\varphi)$.

The alternatives we have considered differ along two dimensions: they can be one- or two-sorted, that is, they have one or two distinct registers of variables and constants; and they may or may not allow an "empty entity". Our results are summarized by the following table:

| | one-sorted | two-sorted |
|---|---|---|
| empty entity | — | SST+ |
| no empty entity | individual mereology, one-sorted plural logic | PFO+ |

In fact, the top left-hand quadrant is populated as well. It is straightforward to modify individual mereology so as to allow an "empty sum", much as SST+ modifies PFO+ by allowing an empty set.

We can provide translations that interpret any one of the theories in any other. Translations that establish the mutual interpretability of the two entries in the right-hand column were sketched in Section 4.1. And translations that establish the mutual interpretability of the two entries in the bottom row were outlined in Section 5.3. Thus, by composing these translations, it follows that any system in the table can be interpreted in any other such system.

As observed in Section 4.2, the existence of these translations and the possibility of interpreting one system in another leave wide open various questions of great philosophical interest. The translations do not necessarily preserve meaning. In fact, the translations may not even preserve truth value on the intended interpretation of the languages in question. Consider a nominalist, who believes that everything is concrete and thus that there are no abstract objects such as sets. This theorist would take various set-theoretic statements to be false although their translations into the plural and mereological idiom are true. Moreover, even philosophers without nominalist scruples will reject as false certain set-theoretic statements whose plural analogues they regard as true. The statement that there is a universal plurality (discussed in Sections 2.6 and 3.5) provides an example. Its translation into ordinary single-sorted set theory is the statement that there is a universal set, which is false according to the standard contemporary conception of set. This apparent mismatch between plural logic and set theory will be a major theme in Chapters 11 and 12.

In the remainder of this chapter, we will consider the relation between pluralities and mereological sums. Does one explain, or even afford an elimination of, the other? Or should both notions be retained? As in the previous chapter, we end up favoring the more liberal option of retaining both notions.

## 5.5 Mereological singularism in linguistic semantics

Mereology is a popular tool among linguists interested in plurals. Indeed, the most influential analysis of plurals in linguistic semantics invokes individual mereology.[7] The popularity of the mereological analysis of plurals is supported by a number of theoretical considerations.[8]

To begin with, mereology provides a framework for the analysis of both plurals and mass terms. The key idea is that plurals are analyzed by means of individual mereology, while mass terms are analyzed by means of material mereology. By appealing to shared mereological structures, one can explain the common features of these two classes of expressions with a high degree of unification. Consider the property of *cumulative reference*. If some people are students and some other people are students, then all of those people are students. Similarly, if some stuff is water and some other stuff is water, then all of that stuff is water. On a mereological analysis, this general phenomenon is captured by assuming that certain properties $P$ "transmit upwards" from the parts to the whole:

$$\forall x \forall y (P(x) \land P(y) \rightarrow P(x + y))$$

Moreover, mass nouns, like plurals, can give rise to collective and distributive readings. Compare:

(5.2) This jewelry is expensive.

(5.3) These pieces of jewelry are expensive.

Both sentences can mean that the jewelry as a whole is expensive. But they can also mean that each piece of jewelry is expensive. A mereological semantics permits a highly unified explanation, for example by assuming

[7] See, e.g., Link 1983, Link 1998, and Champollion 2017. For an overview, see Champollion and Krifka 2016. Alternative approaches based on mereology can be found in Gillon 1992, Moltmann 1997, Landman 2016, and Sutton and Filip 2016.

[8] This section and the next draw from Florio and Nicolas 2020.

that distibutive properties "transmit downwards" from the whole to its salient parts.

Furthermore, there are constructions that combine with plurals and mass nouns but exclude singular count nouns, for instance comparative constructions ('more pieces of jewelry' and 'more jewelry' are grammatical but 'more piece of jewelry' is not) and the proportional quantifier 'most' ('most pieces of jewelry' and 'most jewelry' are grammatical but 'most piece of jewelry' is not). In this case too, one can provide a highly unified analysis by assuming a shared mereological structure of plural and mass nouns.

Another appealing feature of mereology is that it can easily be integrated with the rest of linguistic semantics. Let us explain. In linguistic semantics, one usually interprets natural language by first assigning semantic values to the basic expressions of the language and then deriving the semantic value of more complex expressions compositionally. The stock of available semantic values belongs to a hierarchy generated in the following way. First, one postulates semantic values of some basic types, say objects and truth values. Then, one obtains more semantic values by means of set-theoretic operations applied to the semantic values of the basic types. Any set of objects, for example, is now available as a possible interpretation of a one-place predicate. More generally, the stock of available semantic values may include sets of truth values, and functions between any two sets already available. So the available semantic values inhabit a cumulative hierarchy of sets generated by the entities of the basic types. Mereological sums can be added to the pool of semantic values without fundamentally altering the rest of semantics. These new entities become available for the set-theoretic operations that yield other types of semantic values. The full power of set theory thus becomes available across the semantics. So there is no special difficulty in capturing the fact that plurals, mass terms, and singular count nouns combine in the same way with other grammatical expressions, such as adjectives and verbs, several determiners (for example, 'the', 'some', 'any', and 'no'), and partitive constructions.

Things look different if we try to add pluralities to the stock of semantic values. A plurality is not a special object and hence requires the introduction of a new semantic type. Consider the semantic value of a plural predicate. On the mereological approach, this might be a set of individual sums, where each such sum represents some objects to which the predicate applies. Since the individual sums are objects, they are eligible to figure as elements of a set. Suppose we used primitive plurals instead of individual sums. A plurality is not an object and is thus not eligible to figure as an element of a set, which precludes a set-theoretic representation of the semantic value of a plural predicate. This raises the broader question of how to integrate the new type of primitive plurals with the rest of semantics.

Take the case of 'some'. As shown by the following sentences—all instances of the scheme 'some $\varphi$ $\psi$'—this determiner can combine with singular count nouns, plural count nouns, and mass nouns:

(5.4) Some wolf can be found on the North Pole.

(5.5) Some wolves can be found on the North Pole.

(5.6) Some ice can be found on the North Pole.

The mereological translations of these sentences have the same form:

$$\exists x(\varphi(x) \wedge \psi(x))$$

Each asserts that there is an object that satisfies both $\varphi$ and $\psi$. Thus, on the mereological analysis, the determiner can be seen as making the same semantic contribution in all cases, requiring a common instance of $\varphi$ and $\psi$.

By contrast, these sentences do not have the same representation in plural logic. While (5.4) and (5.6) have the form displayed just above, (5.5) has a different form, namely:

$$\exists xx(\varphi(xx) \wedge \psi(xx))$$

Therefore 'some' appears to have one type of meaning when it combines with a plural count noun and another type of meaning when it combines with a singular count noun or a mass noun.[9]

Thus, we see that linguists have multiple reasons to be attracted to mereological analyses of plurals. Do these analyses have any *philosophical* consequences? Do they reveal, say, how plural talk in natural language should really be understood and thus suggest that plural logic should be eliminated in favor of (individual) mereology? No doubt, the analyses open the possibility of this sort of elimination. But, by themselves, the linguistic reasons for such analyses don't support this philosophical conclusion. Link, however, can be read as suggesting this further, eliminative step:

[9] Another instance of this issue concerns the formulation of a generalized quantifier theory and is discussed in Studd 2015. See Yi 2016 for further discussion.

> While [Boolos] thinks that plural quantification is a self-understood notion I want to argue that this idiom is both in need and capable of a theoretical explanation, which I submit is mereology. (Link 1998, 331–2)

In Section 4.4, we argued that primitive plurals are needed to explain set theory. This argument has an important consequence concerning the possibility of eliminating plurals in favor of (individual) mereology. Plurals provide a more natural basis for the explanation of set theory than mereological sums. For it is more illuminating to explain a set in terms of its many elements than to explain it in terms of the *mereological sum* of these elements. Thus, our argument provides a reason to retain plurals and not eliminate them in favor of mereological sums.

## 5.6 Assessment of singularism in linguistic semantics

The use of individual sums in linguistic semantics requires that we think of sums as objects rather than pluralities. For sums can figure as elements of sets, while pluralities cannot. Therefore mereological talk in linguistic semantics is not one-sorted plural logic in disguise but a genuine form of singularism. As such, it faces the objections already considered in Chapter 3. Our assessment there was that the objections are not compelling, at least not in the absence of substantive assumptions.

In this section, we do two things. First, we discuss a new objection, which has particular force against mereological singularism in the context of linguistic semantics. Then, we revisit one of the substantive assumptions behind some arguments discussed in Chapter 3, namely the possibility of absolute generality. We examine the plausibility of this assumption in the particular context in which we now find ourselves.

The mereological analysis of plurals has raised a concern about the intelligibility of plural predication.[10] Consider the following collective predication:

(5.7) Annie and Bonnie cooperate.

This sentence is perfectly intelligible to competent speakers. According to the mereological analysis, its truth conditions are as follows:

(5.8) 'Annie and Bonnie cooperate' is true if and only if the individual sum denoted by 'Annie and Bonnie' satisfies the predicate 'cooperate'.

[10] See McKay 2006, 24.

However, it may be objected that the right-hand side of (5.8) is unintelligible. We do understand what it is for two people to satisfy the predicate 'cooperate' but—the objection goes—we do not understand what it is for *a sum* to satisfy that predicate.

In response, one may observe that we do understand what it is for a single entity like a group, a team, or a committee to satisfy the predicate 'cooperate'. For (5.9) is perfectly intelligible:

(5.9) This group/team/committee cooperates.

So one may insist that the sense in which a sum satisfies the predicate 'cooperate' is the same sense in which a group, a team, or a committee does.

An alternative response relies on an event-based analysis of predication that generalizes the influential proposal of Davidson 1967. If we broaden the notion of event to include states, we can regard all predicates as properties of events. We can then analyze a sentence like (5.7) in one of two ways.[11]

(5.10) 'Annie and Bonnie cooperate' is true if and only if there is an event of cooperating and the individual sum denoted by 'Annie and Bonnie' is the agent of that event.

(5.11) 'Annie and Bonnie cooperate' is true if and only if there is an event of cooperating, each atom of the individual sum denoted by 'Annie and Bonnie' is a co-agent of that event, and nothing else is a co-agent of that event.

The sole difference concerns the relation between the sum denoted by 'Annie and Bonnie' and the underlying event of cooperating. In the first analysis, the sum is the agent of the event. That is, the sum plays the thematic role of agent of the event. In the second, the sum simply provides the atoms that share the role of agent and, in this sense, function as co-agents of the event. No matter which proposal is adopted, the intelligibility problem should be less pressing: clauses (5.10) and (5.11) appear to be intelligible. The mereological notions involved are given to us through axioms, and we can certainly rely on our ordinary understanding of events for a basic grasp of the event-theoretic notions employed in the semantics. But event semantics is a well-established and successful framework, routinely used by many linguists and

[11] See Landman 2000, Chapter 3, Section 3.2–3.3. For historical details and references, see Oliver and Smiley 2016, 44–5.

philosophers. We see no reason to doubt the coherence of their research and the intelligibility of the event-based analysis of predication.

A theme that emerged in Chapter 3 is that a singularist analysis, such as the mereological one, might not be available in the presence of absolute generality, provided that traditional plural logic is assumed. Let the domain of quantification of our plural object language comprise absolutely everything. We observed in Section 5.2 that mereology can represent the plural only if the objects in the range of the first-order quantifiers are mereological atoms in the individual sense. Since the first-order domain contains absolutely everything, it follows that every object whatsoever is an individual atom. To apply the mereological analysis, however, we would need further objects, namely sums of atoms. Because absolutely every object is now regarded as an atom, no such sums are available. We have, as it were, run out of objects to serve as sums.

How strong is this objection? We will ultimately respond to it by restricting traditional plural logic. But a more immediately appealing response is to deny the possibility of absolute generality. If there is no such thing as absolute generality, then the objection under discussion gets no foothold.

Even if there is some sense to be made of absolute generality—as we argue in Chapter 11—a closely related response nonetheless remains available, namely to observe that, for the vast majority of their purposes, linguists can set aside the problem of absolute generality. They are anyway assuming that the domain is given as a set, for instance when they do generalized quantifier theory. And as we have seen, there is no set of absolutely all objects. Thus, the objection poses no *additional* problem for linguists. Given their purposes, linguists are entitled to proceed precisely as they do.

## 5.7 The elimination of mereology in favor of plural logic

The thesis that mereology should be eliminated in favor of plural logic has found a number of supporters in metaphysics.[12] A systematic development is given by Keith Hossack (2000), who advocates an atomistic metaphysics. According to this view, there really are no complexes such as masses, composite objects, or sets; only metaphysical atoms exist. The view relies essentially on plural logic.

[12] An influential use of this idea is found in van Inwagen 1990; see also Rosen and Dorr 2002.

Hossack points out that none of the usual axioms of mereology, including the ones stated above, seems to hold in general. For example, we can find uses of the word 'part' for which transitivity fails. A page is part of a book, which is part of a library, although the page is not part of the library. And it is highly controversial whether the axiom (M-Sum), which asserts the existence of arbitrary sums, is correct.

According to Hossack, "[a]bout the only interpretation on which the mereological axioms are indisputable logical truths is a plural one" (Hossack 2000, 423). The formal translation from mereology to one-sorted plural logic can be seen as vindicating this point. Indeed, he gestures at the result and concludes that:

> it seems plausible that we can use the *are-some-of* relation to give an analysis of our ordinary talk of parts and wholes that is superior to the account given by extensional mereology. (Hossack 2000, 424)

Finally, he shows how the analysis can be carried out for various complexes.

Simplifying a bit, the proposed strategy is illustrated by the following examples of elimination concerning masses and complex objects.

(5.12) There is some water.

(5.13) Some atoms are $\varphi$.

(5.14) There is a chair.

(5.15) Some atoms are $\psi$.

Here $\varphi$ and $\psi$ are collective predicates true of atoms that constitute water and atoms that constitute a chair, respectively. In the literature, the latter is usually rendered as "are arranged chairwise".

There are three main issues with this approach. First, what guarantee do we have that all composite objects decompose into atoms? Aristotle famously held that matter is indefinitely divisible. Any bit of matter contains an even smaller bit of matter. Whether or not he was right about that, it certainly seems possible that there could be atomless gunk, that is, some stuff without atomic parts.[13] Thus, the proposed analysis depends on a risky and controversial metaphysical assumption.

[13] More formally, $x$ consists of atomless gunk if and only if any part $y$ of $x$ has a proper part $z$.

Second, how should we analyze plural talk about composite objects?[14] Consider the following collective predication about a plurality of composite objects:

(5.16) The chairs are arranged in a circle.

The problem is that talk about a single chair already uses plurals, in the form of plural talk about some atoms arranged “chairwise”. So we have already “used up” the plural resources of the language in which we give our analysis. Pluralities of composite objects would therefore require superplurals. (We discuss the legitimacy of superplurals in Chapter 9.)

Finally, there appears to be a mismatch between the modal profiles of a plurality and that of a composite object. Plural membership is modally rigid. If *a* is one of *bb*, then necessarily so (at least on the assumption that all of the objects in question continue to exist). And likewise for not being one of some things. In short, a plurality doesn't vary with respect to which members it has in different circumstances or possible worlds. (This view is defended in Chapter 10.) By contrast, there are composite objects for which parthood appears non-rigid. Consider one of your cells. It seems possible for you to exist even though this cell is no longer to be part of you. And a good thing too, since the life expectancy of most cells is far shorter than that of the organism to which they belong![15]

## 5.8 Keeping both plural logic and mereology

Where does this leave us? We argued in Chapter 4 that both pluralities and sets should be retained. Should mereological sums too be retained alongside pluralities and sets?

Our previous discussion suggests an “algebraic conception” of mereology. The axioms of mereology describe a certain kind of abstract structure, which can be realized in many different—indeed non-isomorphic—ways.[16]

[14] For a discussion of this objection, see Uzquiano 2004a.

[15] Even if our claims about a mismatch of modal profiles is right, this isn't the end of the story—as so often in philosophy. The mismatch is analogous to that of the modal properties of the statue and the clay in the famous problem of material constitution. In both cases, a proponent of the relevant reduction can attempt to address the mismatch by invoking a counterpart relation (see Lewis 1971 and Gibbard 1975).

[16] See Fine 2010, Section II, for a similar view based on a “pluralist” conception of parthood.

We have seen various examples of such realizations: the material interpretation, where $x$ is part of $y$ just in case the matter of $x$ is contained in that of $y$; and the plural interpretation in the one-sorted formulation of plural logic. This suggests that mereology, unlike set theory, does not have a single canonical interpretation. Mereology is the abstract theory of part-whole structures, which are realized in many different ways. In this respect, mereology is rather like the theory of partial orders. It makes no sense to ask what is the true partial order of reality. A plethora of different partial orders are realized throughout reality. Likewise, we submit, it makes no sense to ask what is the true part-whole structure of reality. There are many such structures.

The question, then, is: what part-whole (or mereological) structures are there? We have already mentioned two examples: material parthood and the among-relation defined on pluralities. For our purposes, the most important aspect of the question concerns individual mereology, which as we have seen plays a key role in many linguistic approaches to plurals. Are there individual sums?

This metaphysical question has no easy answer. A comprehensive discussion would take us too far afield. Instead, we will briefly present two reasons to accept the existence of individual sums. First, individual sums are very useful in semantics in order to account for various natural language phenomena. This provides a broadly naturalistic reason to accept them, namely that individual sums figure in respectable scientific practice.

Second, as explained in Section 4.4, we are attracted to a liberal view of definitions. According to this view, it suffices for a mathematical object to exist that an adequate definition of it can be provided, where the adequacy in question is understood as follows. Suppose we start with a domain of objects standing in certain relations and would like to define one or more additional objects. Suppose our definition determines the truth of any atomic statement concerned with the desired "new" objects by means of some statement concerned solely with the "old" objects with which we began. Then, according to our liberal view, the definition is permissible.

Let us apply this approach to our question about the existence of individual sums. Suppose we start with some domain of objects. For every plurality of objects $xx$ from this domain, we postulate their individual sum $\Sigma(xx)$, which contains each member of $xx$ as an individual atom. Atomic predications concerned with these objects are to be assessed as follows.

(a) If $xx$ consist of just a single object $y$, then the sum $\Sigma(xx)$ is identical with $y$.
(b) $\Sigma(xx) \leq \Sigma(yy)$ if and only if $xx \preccurlyeq yy$.

Clearly, (b) entails:

(c) $\Sigma(xx) = \Sigma(yy)$ if and only if $xx \approx yy$.

This yields an account of the desired individual sums and their relations in terms of pluralities of the objects with which we started and *their* relations.

The account of individual sums clashes with certain metaphysicians' attempts to eliminate mereology in favor of pluralities. The result of adopting the liberal view of definitions yields mereological sums as objects, much as the application of Gödel's "set of" operation to pluralities yields sets as objects. These objects live alongside the pluralities from which they are formed. By contrast, the *eliminative* project surveyed in Section 5.7 rejects the existence of all mereological non-atoms. Overall, our view is that we should retain two kinds of derived objects—sets and individual mereological sums—both of which can be accounted for in terms of pluralities. Because of this account, our view is a form of *non-eliminative* reductionism.

We have advocated retaining pluralities, sets, and mereological sums. How do these three kinds of object interact? This question raises a number of interesting and difficult issues. We shall content ourselves with commenting on one particularly important point. How does the individual sum $\Sigma(xx)$ differ from the set $\{xx\}$? Part of the answer has to do with clause (a): while the sum of a singleton plurality is identical with the sole member of this plurality, the set formed by a singleton plurality is distinct from its sole member. Another part of the answer emerges when sum formation is iterated. Clause (b) must then be replaced by a more general criterion of identity. Let '$z \circ xx$' abbreviate $\exists x(z \circ x \wedge x \prec xx)$. Then this more general criterion can be formulated as:

$$(\mathrm{b}^+) \qquad \Sigma(xx) \leq \Sigma(yy) \;\leftrightarrow\; \forall z(z \circ xx \rightarrow z \circ yy)$$

which, unlike (b), is valid even when $xx$ and $yy$ are not all individual atoms. Clearly, ($\mathrm{b}^+$) entails:

$$(\mathrm{c}^+) \qquad \Sigma(xx) = \Sigma(yy) \;\leftrightarrow\; \forall z(z \circ xx \leftrightarrow z \circ yy)$$

These clauses show that sum formation is "flat" in a way that set formation is not.[17] That is, taking the sum of some objects and some other objects is the same as taking the sum of the former objects and *the sum of* the latter objects. The analogous set-theoretic claim is false: taking the set of some objects and some other objects is not the same as taking the set of the former objects and *the set of* the latter objects. To be precise, let us formalize these observations. Let $t_1$ and $t_2$ be two terms, either singular or plural, and let $tt$ be the plural term referring to all of the objects referred to by either $t_1$ or $t_2$. Using $\Sigma(t_1, t_2)$ as a shorthand for $\Sigma(tt)$, we then have that $\Sigma(xx, \Sigma(yy)) = \Sigma(xx, yy)$, while the analogous set-theoretic claim, $\{xx, \{yy\}\} = \{xx, yy\}$, is false. Thus, as advertised, sums and sets behave in importantly different ways.

[17] Kit Fine develops a similar but more general view. His "sums" (2010, 574) correspond to our individual sums.

# Appendices

## 5.A Partial orders and principles of decomposition

The appendices to this chapter have two main aims. First, we want to provide a useful introduction to mereology. We begin with the axioms of the atomistic version of Classical Extensional Mereology, mentioned in Section 5.1. We present the axioms in natural groups, where each group captures one fairly unified idea. Second, based on the resulting understanding of mereology, we prove the mutual interpretability of our official two-sorted plural logic PFO+ and the appealing one-sorted alternative based on the described mereological theory.

We begin by rehearsing some definitions. Let '$\leq$' be an atomic predicate representing 'is part of (or equal to)'. Then we make the following definitions.

**Definition 5.1 (Basic notions)**

(a) $x < y$ ($x$ is a *proper part* of $y$) iff $x \leq y \wedge x \neq y$.
(b) $x \circ y$ ($x$ *overlaps* $y$) iff $\exists z(z \leq x \wedge z \leq y)$.
(c) $x \perp y$ ($x$ *is disjoint from* $y$) iff $\neg x \circ y$.
(d) $\mathrm{At}(x)$ ($x$ is *an atom*) iff $\neg\exists y(y < x)$.

The first group of axioms, which is already familiar, consists of those of a partial order.

**Definition 5.2 (Partial order)** $\leq$ is a *partial order* iff:

(PO1) $$x \leq x$$

(PO2) $$x \leq y \wedge y \leq x \rightarrow x = y$$

(PO3) $$x \leq y \wedge y \leq z \rightarrow x \leq z$$

Let *PO* be the first-order theory whose axioms are (PO1)–(PO3).[18]

The second group of axioms are principles of decomposition. They sanction that the mereological relations that obtain between two objects are a matter of these objects' parts. First, there are the *supplementation axioms*:

[18] In the statement of these axioms, we rely on our convention from Section 2.4 of omitting initial universal quantifiers, as is often done in mathematical prose.

(WS) $x < y \to \exists z(z \leq y \land z \perp x)$ (Weak Supplementation)

(SS) $x \nleq y \to \exists z(z \leq x \land z \perp y)$ (Strong Supplementation)

Next, there is the principle of complementation:

(C) $x \nleq y \to \exists z \forall w(w \leq z \leftrightarrow w \leq x \land w \perp y)$ (Complementation)

If $x \nleq y$, the object $z$ said to exist by (C) is easily seen to be unique; this object is often written '$x \setminus y$' (pronounced "$x$ minus $y$").[19]

Our first result orders the principles of decomposition by their logical strength. Let us say that $\varphi$ is *strictly stronger than* $\psi$ *relative to a theory* $T$ iff $T, \varphi \vdash \psi$ but $T, \psi \nvdash \varphi$. Then:

**Lemma 5.1** Relative to the theory PO of partial orders, we have: (C) is strictly stronger than (SS), which is strictly stronger than (WS), which is strictly stronger than just PO.

*Proof.* The implications are straightforward. First, $x \setminus y$ can serve as the object $z$ said to exist by (SS). Second, we use the fact that the definition of $x < y$ assures $y \nleq x$. The three non-implications are established by means of counterexamples. See Varzi 2019, Section 3, for details. ⊣

Strong Supplementation is particularly important because it ensures that parthood admits of a very useful characterization in terms of overlap, namely:

(∗) $$\forall z(z \circ x \to z \circ y) \;\leftrightarrow\; x \leq y$$

Let us call $\forall z(z \circ x \to z \circ y)$ *the overlap criterion* for the parthood claim $x \leq y$. Thus, (∗) asserts the validity of the overlap criterion for parthood. Our next result reveals the tight connection between Strong Supplementation and the validity of the overlap criterion.

**Lemma 5.2** (SS) is equivalent to (∗) relative to the theory PO.

*Proof.* First, observe that some simple first-order logic allows us to rewrite (SS) as:

(SS′) $$\forall z(z \leq x \to z \circ y) \to x \leq y$$

[19] To prove uniqueness, assume there were two such objects, $z_1$ and $z_2$. Then we would have $z_1 \leq z_2$ and $z_2 \leq z_1$, whence $z_1 = z_2$ after all.

Next, PO proves the equivalence of $\forall z(z \leq x \rightarrow z \circ y)$ and $\forall z(z \circ x \rightarrow z \circ y)$. It follows that (SS) is equivalent, relative to PO, to the left-to-right direction of $(*)$. Our claim therefore follows because the other direction of $(*)$ is a theorem of PO. ⊣

Strong Supplementation has another attractive consequence as well, which is recorded in the following lemma.

**Lemma 5.3** The following statements are equivalent relative to PO plus (SS):

(i) $x = y$
(ii) $\forall z(z \leq x \leftrightarrow z \leq y)$
(iii) $\forall z(z \circ x \leftrightarrow z \circ y)$
(iv) $\forall z(z \perp x \leftrightarrow z \perp y)$

*Proof.* Relative to PO, (i) implies (ii), which in turn implies (iii). We now use Lemma 5.2 to establish that (iii) implies (i) relative to PO + (SS). Thus, the first three conditions are equivalent. Finally, we observe that (iii) is equivalent to (iv) because $z \perp x \leftrightarrow z \perp y$ can be rewritten as $\neg z \circ x \leftrightarrow \neg z \circ y$. ⊣

## 5.B Some notions of sum

We now describe two conceptually different notions of sum that are available in the context of any partial order $\leq$.

The first notion is that of a *least upper bound*. Let us say that $z$ is an *upper bound of $x$ and $y$* iff $x \leq z$ and $y \leq z$. A *least upper bound of $x$ and $y$* is an upper bound $z$ of $x$ and $y$ such that, for any other upper bound $w$, we have $z \leq w$. The statement that $z$ is a least upper bound of $x$ and $y$ can be formalized as:

$$\forall w(z \leq w \leftrightarrow x \leq w \wedge y \leq w) \tag{5.17}$$

Clearly, when a least upper bound of two objects exists, it is unique.[20]

A second notion of sum is defined in terms of the notion of overlap, namely that $z$ is *a fusion of $x$ and $y$* iff:

$$\forall w(w \circ z \leftrightarrow w \circ x \vee w \circ y) \tag{5.18}$$

[20] Suppose both $z_1$ and $z_2$ were least upper bounds of $x$ and $y$. Then we would have $z_1 \leq z_2$ and $z_2 \leq z_1$, which entails $z_1 = z_2$.

That is, a fusion of $x$ and $y$ is an object $z$ such that, to overlap $z$ is equivalent to overlapping either $x$ or $y$. Assume Strong Supplementation. Then, if there is a fusion of $x$ and $y$, this fusion is unique. To see this, suppose that $z_1$ and $z_2$ are fusions of $x$ and $y$. By our definition of a fusion, an object $w$ overlaps $z_1$ iff $w$ overlaps $z_2$ (namely, iff $w$ overlaps either $x$ or $y$). By the overlap criterion of identity—which by Lemma 5.2 is available on the assumption of PO and Strong Supplementation—it follows that $z_1 = z_2$.

What is the relation between the two notions of sum? The next result provides the answer.

**Lemma 5.4**

(a) Assume Strong Supplementation. Then any fusion of $x$ and $y$ is also a least upper bound of $x$ and $y$.

(b) Assume Complementation. Then any least upper bound of $x$ and $y$ is also a fusion of $x$ and $y$.

*Proof.* (a) Assume $z$ is a fusion of $x$ and $y$, that is, (5.18). The overlap criterion for parthood immediately yields $x, y \leq z$. It remains to show that $z$ is the least upper bound of $x$ and $y$. So assume $x, y \leq z'$. This assumption, combined with (5.18), yields $\forall w(w \circ z \rightarrow w \circ z')$, whence by the overlap criterion again, we have $z \leq z'$, as desired.

(b) Assume $z$ is a least upper bound of $x$ and $y$, that is, (5.17). Because $x, y \leq z$, we have that $w \circ x \vee w \circ y$ implies $w \circ z$. It remains to establish the converse implication. Assume, for contradiction, that $w \circ z$ but $w \perp x \wedge w \perp y$. This means that $z \nleq w$, whence by Complementation, we can let $u$ be $z \setminus w$, that is:

(5.19) $$\forall v(v \leq u \leftrightarrow v \leq z \wedge v \perp w)$$

Instantiating '$\forall v$' with respect to $x$ and $y$, it follows that $x, y \leq u$. But $w \circ z$, so $u < z$. This result contradicts our assumption that $z$ is *the least* upper bound of $x$ and $y$. That establishes the implication we set out to prove. ⊣

We now formulate axioms stating that *any two objects* have a sum in each of our two senses:

(LUB) $$\exists z(\forall w(z \leq w \leftrightarrow x \leq w \wedge y \leq w))$$

(Fus) $$\exists z(\forall w(w \circ z \leftrightarrow w \circ x \vee w \circ y))$$

What about *larger collections of objects*: do these too have sums? Let us start with sums understood as least upper bounds. A partial order $\leq$ is said to be *complete* iff, for any instantiated condition $\varphi(x)$, there is a least upper bound of all the objects that satisfy the condition; that is, iff:

$$\text{(LUB}^+\text{)} \qquad \exists x\, \varphi(x) \rightarrow \exists z \forall w (z \leq w \leftrightarrow \forall x(\varphi(x) \rightarrow x \leq w))$$

Next, the partial order is said to *permit unrestricted fusion* iff the following axiom scheme holds:

$$\text{(Fus}^+\text{)} \qquad \exists x\, \varphi(x) \rightarrow \exists z \forall w (w \circ z \leftrightarrow \exists x(\varphi(x) \wedge w \circ x))$$

This is meant to capture the idea that any non-empty collection of objects has a fusion.

As before, and under the same assumptions as before, we can prove that a least upper bound (or fusion), if there is one, is unique. Also as before, and under the same assumptions as before, we can prove that these unrestricted notions of least upper bound and fusion are equivalent.

Equipped with this understanding of unrestricted sums, we are now ready to define Classical Extensional Mereology.

**Definition 5.3** The theory of *Classical Extensional Mereology* consists of:

(i) the theory PO of partial orders
(ii) the axiom of Strong Supplementation
(iii) the axiom scheme (Fus$^+$)

Various alternative (but equivalent) axiomatizations exist as well. Here is one important example: we may replace (ii) and (iii) with the axiom of Complementation and the axiom scheme (LUB$^+$), respectively.[21]

[21] To see that this alternative yields an equivalent axiomatization, we first invoke the mentioned generalization of Lemma 5.4. It then remains only to show that Classical Extensional Mereology, as defined above, entails Complementation. (Hint: Show that $x \setminus y$ can be defined as the fusion of $w$ such that $w \leq x \wedge w \perp y$.) See Varzi 2019, Section 4.4, for a useful overview of further alternatives, including more minimalistic ones. The articulation of the axioms of Classical Extensional Mereology into the three groups (i) through (iii) is nevertheless historically important and, we think, conceptually more illuminating than the more minimalistic alternatives.

## 5.C Atomicity

**Definition 5.4** A partial order $\leq$ is said to be *atomistic* iff every object has an atomic part:

(At) $$\forall x(\exists u(u \leq x \wedge \mathrm{At}(u))$$

Let *Atomistic Classical Extensional Mereology* be the result of adding (At) to Classical Extensional Mereology.

**Lemma 5.5** Let $\leq$ be an atomistic partial order with Strong Supplementation.

(a) Then parthood can be tested on atoms, in the following sense:

$$x \leq y \leftrightarrow \forall u(\mathrm{At}(u) \rightarrow (u \leq x \rightarrow u \leq y))$$

(b) Assume $\leq$ is also complete; that is, (LUB$^+$) holds. Then every object is is identical to the fusion of its atoms.

*Proof sketch.* The proof of (a) is routine and is therefore omitted. For (b), consider any object $x$, and let $y$ the fusion of the atoms in $x$. By Atomicity, anything that overlaps $x$ is easily shown also to overlap $y$. Moreover, anything that overlaps $y$ overlaps an atom in $x$ and thus also $x$ itself. By the overlap criterion of identity, it follows that $x = y$. ⊣

Our main reason for being interested in atomistic mereology is that the among-relation $\preccurlyeq$ is atomistic. Recall that this relation is defined in the two-sorted system, but primitive in the one-sorted system. Consider the plural comprehension scheme:

(P-Comp) $$\exists x\, \varphi(x) \rightarrow \exists xx \forall x(x \prec xx \leftrightarrow \varphi(x))$$

The analogue of this principle in the atomistic mereology of $\preccurlyeq$ is the principle stating that, provided $\varphi$ is instantiated by an atom, there is a sum whose atomic parts are all and only the atomic $\varphi$s:

(At-Sum) $$\exists x(\mathrm{At}(x) \wedge \varphi(x)) \rightarrow \exists z \forall w(\mathrm{At}(w) \rightarrow (w \leq z \leftrightarrow \varphi(w)))$$

Let us now compare this atomistic principle with our unrestricted fusion principle ($\text{Fus}^+$).

**Theorem 5.1** ($\text{Fus}^+$) is strictly stronger than (At-Sum) relative to the theory of partial orders with Strong Supplementation. However, if we add the assumption that the partial order is atomistic, the two principles become equivalent.

*Proof.* Assume $\exists x(\text{At}(x) \wedge \varphi(x))$. Consider the fusion of all atomic $\varphi$s. This fusion is easily shown to be a witness for the existential claim in the consequent of (At-Sum). The converse implication is easily seen to fail when $\leq$ is non-atomic. Finally, assume that $\leq$ is atomistic. Suppose there is a $\varphi$. We want to show that there is a fusion of all $\varphi$s. Let $z$ be the sum of all atomic parts of $\varphi$s, which is ensured by (At-Sum) to exist. That is, we instantiate (At-Sum) with the formula '$\exists y(\varphi(y) \wedge x \leq y)$'. Then, since $\leq$ is atomistic, to overlap $z$ is equivalent to overlapping some atomic part of some $\varphi$. And to overlap some atomic part of some $\varphi$ is (again, since $\leq$ is atomistic) equivalent to overlapping some $\varphi$. By transitivity, to overlap $z$ is equivalent to overlapping some $\varphi$. Thus, $z$ is our desired fusion of all $\varphi$s. $\dashv$

## 5.D One- and two-sorted plural logic compared

Let us compare the one- and two-sorted formulations of plural logic. The former, we recall from Section 5.3, states that the among relation $\preccurlyeq$ satisfies the axioms of Atomistic Classical Extensional Mereology. More precisely, this theory is just like Atomistic Classical Extensional Mereology, as formulated above, except that it uses the predicate '$\preccurlyeq$' rather than '$\leq$' and that all of its variables are doubled (for example, '$xx$' instead of '$x$'). The latter is the familiar system PFO+. We also provided translations from each language into the other. Let us now prove the promised result that these translations are interpretations of each formulation of plural logic in the other.

Consider first the result of translating the system PFO+ into one-sorted plural logic. The axioms and inference rules of PFO+ are easily seen to be mapped to theorems and derived rules of one-sorted plural logic. First, the axioms and rules of sentential logic and the quantifiers rules are straightforward. Next, consider the indiscernibility principle:

(Indisc) $\qquad xx \approx yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))$

The antecedent is translated as the statement that $xx$ and $yy$ have the same atomic parts. By Lemma 5.5(a) and the assumption that $\preccurlyeq$ is a partial order, we derive the identity statement $xx = yy$, whence the translation of the consequent of (Indisc) follows by Leibniz's Law.

Then, there is the axiom which says that every plurality is non-empty:

(Non-Empty) $$\forall xx \exists y (y \prec xx)$$

This is mapped to the axiom that states that the order $\preccurlyeq$ is atomistic.

Finally, each instance of the plural comprehension scheme (P-Comp) is mapped to an instance of the atomistic principle (At-Sum), which is a theorem of our one-sorted plural logic by the first half of Theorem 5.1.

We turn now to the reverse direction, namely the interpretation of one-sorted plural logic in PFO+. As mentioned in Section 5.3, the primitive among-relation $\preccurlyeq$ of one-sorted plural logic is translated as its defined counterpart in PFO+:

(5.20) $$xx \preccurlyeq yy \leftrightarrow_{\text{def}} \forall u (u \prec xx \rightarrow u \prec yy)$$

So we must verify that this defined relation satisfies the required properties.

**Theorem 5.2** The defined relation $\preccurlyeq$ in PFO+ satisfies the axioms of Atomistic Classical Extensional Mereology.

*Proof.* It is immediate from its definition that the relation $\preccurlyeq$ is a partial ordering. Using (P-Comp) we can easily derive Strong Supplementation and, via (Non-Empty), Atomicity. The completeness axiom ($\text{Fus}^+$) also follows from an appropriate use of (P-Comp). That is, an application of (P-Comp) to the formula '$\exists xx(\varphi(xx) \wedge u \prec xx)$' delivers the fusion of all pluralities which satisfy $\varphi$. (This is, essentially, the second half of Theorem 5.1.) ⊣

Putting everything together, we obtain our main result.

**Theorem 5.3** The two-sorted system PFO+ and the one-sorted plural logic are mutually interpretable.