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Poincaré and the Prehistory of Mathematical Structuralism

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1. Introduction and Historical Background

"Structuralism" denotes a family of views united by a common conception of the subject matter of mathematics. According to this approach, mathematics is not "about" mathematical objects, such as the number 2; nor is it even about specific mathematical systems, such as the Dedekind-Peano natural numbers. Rather, mathematics is about something more abstract: mathematical structure.

As a fully-fledged philosophy of mathematics, structuralism is young. Its birth is associated with Benacerraf's famous "What Numbers Could Not Be," and its current form has been shaped by subsequent work by Hellman, Resnik, Shapiro, and others.[1] Well before any of this recent philosophical work, however, a more general structuralist conception emerged from mathematical practice, that is, from mathematicians reflecting on their methodology and subject matter. We can call these earlier views "methodological" structuralism (Reck and Price 2000). Some questions about methodological structuralism include the following. How far into past mathematics does it go? How does philosophical structuralism arise from it? Is there really a sharp distinction between methodological and philosophical structuralism?

For this chapter I take for granted that one can distinguish methodological structuralism from further philosophical views about mathematical structures. The former is a simpler, working view about the general subject matter and methodology of mathematics, independent of any specific metaphysical and/or epistemological views about structures. For example, it is a commonplace that the development of non-Euclidean systems made geometry more abstract: the subject matter ascended to a more general perspective, to accommodate multiple geometric systems. Basic methodological structuralism solves any concern about this change by viewing geometry as the study of (possible) geometric structure

[1] See Benacerraf (1965), Hellman (1989), Resnik (1997), and Shapiro (1997) for a start.

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rather than as the attempt to provide a (true) theory of space. Another example concerns the way symbolical algebra developed out of initiatives for teaching calculus in Britain in the 19th century. Detaching the calculus symbols from any particular interpretation is a move toward abstraction that, again, fits well with a structuralist view of the subject matter. In both of these cases, however, neither the shifting subject matter nor its mathematical treatment depended on or derived from philosophical structuralism, such as specific views about the metaphysical nature of structures.

That said, it seems clear that methodological structuralism has been in the air for quite some time. Indeed, some of the success of key historical mathematical figures can be correlated with this new way of thinking about their subject matter. For example, Hankel writes in 1867 that mathematics is

> purely intellectual, a pure theory of forms, which has for its object not the combination of quantities or their images, the numbers, but things of thought to which there could correspond effective objects or relations, even though such a correspondence is not necessary. (Kline 1972, 1031)

Earlier, Gauss articulates a similar view:

> One quantity in itself cannot be the object of a mathematical investigation; mathematics considers quantities only in their relation to one another. . . . Now, mathematics really teaches general truths concerning the relations of quantities. (Gauss 1829, paragraphs 2, 3)

And even earlier, in the mid-18th century, both D'Alembert and Maclaurin express similar ideas in defending the calculus.[2] Both emphasize the *method* of calculus to justify its subject matter rather than vice versa (as, for example, Berkeley [1734] appears to have demanded). And the method highlights relations—supported by the clear conception of, and evidence for, those relations—over the existence and nature of specific types of objects. The term "methodological" structuralism is thus apt.

One might object that emphasizing mathematical relations is not structuralism, since relations are too specific. For example, though the "greater than" relation between numbers is different from the "older than" relation between

[2] For example, Maclaurin writes: "The mathematical sciences treat of the relations of quantities to each other, and of . . . every thing of this nature that is susceptible of a regular determination. We enquire into the relations of things, rather than their inward essences, in these sciences. . . . It is not necessary that the objects of the speculative parts should be actually described, or exist without the mind; but it is essential, that their relations should be clearly conceived, and evidently deduced" (1742, Ewald, 116; 51 in original).

possible physical objects, such differences are beside the point for a structuralist. Structuralism abstracts not only from the referents of singular terms (objects) but also from the meanings of relational terms and properties. So an emphasis on mathematical relations does not add up to structuralism.

However, both emphasizing relations over objects, and focusing on method over content, reference, and meaning, are important steps toward a more robust structuralism. The idea that a theory can be grounded in *what it does* rather than *what it is about* is significant, for it naturally leads to the view that mathematics need not have any particular (object level) subject matter. Finally, some (also beyond Hankel)[3] explicitly connect the relational nature of mathematics to the view that it is about "form" rather than content.[4]

I cannot argue for these general claims, nor am I attempting to answer a historical question with any precision. Whether a cause or a response, methodological structuralism emerges from a close connection to mathematical practice; and it goes back at least to early defenses of the calculus. So it was clearly in the air before Poincaré.[5]

Like many others, Poincaré expresses this basic structuralist view:

> Mathematicians study not objects, but relations between objects; the replacement of these objects by others is therefore indifferent to them, provided the relations do not change. The matter is for them unimportant, the form alone interests them. ([1902] 1952, chap. 2, 44)

His point concerns the subject matter of mathematics—that which mathematicians "study." The emphasis on form as well as relations justifies classifying him as at least a basic methodological structuralist.

As we'll see, however, Poincaré professes further philosophical views about the *nature* of mathematical structures. These are clearest in his remarks about groups and the group concept. I will argue that his conception of mathematical intuition can be understood similarly. That is, Poincaré's views about the nature and knowledge of mathematical structure are philosophically significant, extending beyond methodological structuralism.

[3] Compare, e.g., the article on Grassmann by Paola Cantù in the present volume.

[4] For example, Gauss's emphasis on relations between *quantities* might seem to cast him as a mere pre-structuralist. But Gauss also writes in defense of complex numbers: "The mathematician abstracts totally from the nature of the objects and the content of their relations; he is concerned solely with the counting and comparison of the relations among themselves" (1831, paragraph 22). Gauss thus emphasizes abstraction generally—from relations as well as objects. Mathematics is about neither individual objects nor particular relations; it makes comparisons between relations, yielding insights into relation-forms. The view is thus genuinely structuralist. For more on Gauss, see the chapter by Ferreirós and Reck in the present volume.

[5] Others whose work seems in the structuralist spirit around this time include Serret (1819–1885), De Morgan (1806–1871), and Galois (1811–1832).

I will proceed as follows. In section 1, I focus on Poincaré after briefly sketching some basics on structuralism in the philosophical literature. I aim to show that Poincaré endorses structuralist ideas, arguing further that in some ways his view best aligns with a "strong" version of structuralism, the "structure first" view. In section 2 I address the question of how extensive this strong view is and whether it is consistent with some of Poincaré's other philosophical commitments. Thus, before concluding, I consider his semi-Kantian views about mathematical intuition, his structural realist views in natural science, and whether they can all be consistently combined with the strong, "structure first" view about some aspects of mathematics.

2. Structuralism: A Basic Taxonomy and Poincaré's Place in It

2.1. Some Basic Types

Structuralism is the view that mathematics is about abstract structures rather than specific mathematical objects or even specific systems of objects. For example, consider the natural number 3 as defined by Zermelo. Contrast this with the set of Zermelo natural numbers, and also with the natural number structure. The latter can be thought of as the form of all systems of natural numbers, regardless of how the particular systems and objects are defined or construed. The natural number structure is what the Zermelo numbers, the von Neumann numbers, and the Frege numbers have in common. Structuralism is the view that mathematics is about this sort of thing.

As noted, this view about the subject matter of mathematics does not entail any specific metaphysical views about the *nature* of that subject matter, nor about how we *know* mathematical structures. Thus, the basic structuralism/nonstructuralism distinction is different from realism/anti-realism disputes about the (independent) existence of mathematical objects. A structuralist may or may not think mathematical objects exist independently of mathematicians; she also may or may not think structures so exist. One can also be a realist, or Platonist, about structures, believing that they exist independent of the minds/constructions of human mathematicians. Alternatively one can be a Platonist about mathematical objects and systems, but not abstract structures. One can of course also be anti-realist about all abstract objects. So basic structuralism is simply a view about the subject matter of mathematics, remaining neutral about the nature of structures.[6]

[6] Independent or dependent, and if the latter, dependent *on what*.

Philosophical structuralism, on the other hand, aims to articulate and defend some of these further properties. It is important to note, however, that the differences between structuralist philosophies are generally not explained in terms of the familiar issue of the dependence or independence of mathematical objects/reality on *mathematicians* (and their constructions, proofs, etc.). Instead, these further views are typically characterized in terms of the relationship between mathematical structures on the one hand and the mathematical *objects/systems* that instantiate them on the other. That is, people don't play any immediate role in a typical basic structuralist taxonomy.

For example, Shapiro explains three main structuralist views as follows:

> Any of the usual array of philosophical views on universals can be adapted to structures. One can be a Platonic *ante rem* realist, holding that each structure exists and has its properties independent of any systems that have that structure. On this view, structures exist objectively, and are ontologically prior to any systems that have them (or at least ontologically independent of such systems). Or one can be an Aristotelian *in re* realist, holding that structures exist, but insisting that they are ontologically posterior to the systems that instantiate them. Destroy all the natural number systems and, alas, you have destroyed the natural number structure itself. A third option is to deny that structures exist at all. Talk of structures is just a convenient shorthand for talk of systems that have a certain similarity. (Shapiro, n.d., Part 1)

Like ordinary Platonism, Platonic *ante rem* structuralism espouses a kind of realism about structures in that the structures are independent of both objects and systems of objects for their existence. So *ante rem* structuralism, one might say, simply adds another kind of universal to the old Platonic universe: mathematical structure.

A common analogy to explain this view involves the distinction between places or offices and the objects that can occupy those places. For example, reference to the US president might be to a particular person, as in "The president is tired." But it may also refer to the position independent of who occupies it, as in "The president heads the executive branch of the government."

With this distinction in mind, *ante rem* structuralists generally view mathematical assertions as more like the latter than the former—as assertions about offices rather than occupants of those offices. Furthermore, the truth-value of such assertions is held to be indifferent to whether or not the places in the structure *have* occupants. So, for example, arithmetic studies the natural number structure, which exists independently of any individual natural numbers as well as any particular systems (definitions) of the numbers. For the Platonic *ante rem* structuralist, abstract structures are what concern mathematicians.

Although Shapiro considers *in re* structuralism another (though weaker) form of realism about structures, it also bears some similarity to ordinary constructivism in the philosophy of mathematics. Constructivists think mathematical objects exist, but only dependently—on the constructions carried out by mathematicians. Similarly, on the *in re* view, structures exist, but only dependently—on the existence of systems of mathematical objects instantiating those structures. (So systems rather than human constructions.) Whereas the *ante rem* view asserts the ontological *priority* and *independence* of structures from objects and systems, the *in re* view asserts the ontological *posteriority* of structures and their *dependence* on mathematical objects/systems. For example, the *in re* view is that "2 + 3 = 5" is a truth about the natural number structure because it is true of *any occupants* of the "offices" 2, 3, and 5.

The third main option, eliminativism (e.g., fictionalism and modal structuralism), is a form of anti-realism, or nominalism, about structures. On this view, structures don't actually exist. Whether or not mathematical objects or systems exist independently of mathematicians, talk of mathematical structure is simply a convenient way to speak.

The point for us is that these possible views about structure are differentiated with respect to underlying mathematical objects/systems, rather than mathematicians and their activities. Issues of dependence or independence thus do not correspond to the ordinary Platonism-constructivism debates in the philosophy of mathematics. Even the eliminativist option is expressed as an anti-realism or constructivism about structure only; it appears that one could be fictionalist about structure and realist about particular mathematical systems or objects. With this in mind I will argue that in this taxonomy, Poincaré's views about mathematical structure most closely match Shapiro's *ante rem* category. With the *ante rem* structuralist, Poincaré advocates the priority and independence of some structures to their systems—despite the fact that he is a constructivist, not a Platonist, about mathematics. This "structure first" view is why I consider him as holding a position one might call "constructivist *ante rem* structuralism." (I will return to the apparent incongruity of this position, in section 2.)[7]

[7] Because of this interpretation, my argument will involve pointing out that the taxonomy referenced here is incomplete. (This is not a criticism; Shapiro did not claim to provide a complete taxonomy.) As noted, since the issues of priority and independence are articulated relative to other mathematical objects, they don't engage in the ordinary discourse regarding realism versus constructivism. In particular, the priority of structures over systems seems detachable from metaphysical realism; that is, one need not be a Platonist to endorse the *ante rem* relationship between structures and systems.

2.2. Poincaré as a Structuralist

An important mathematician during a significant time (1854–1912), Poincaré reflected on the increasing abstraction of mathematics and its impact on both the subject matter and our knowledge of it. Structure is a central concept guiding his understanding of these changes. The two main purposes of this section are (i) to clarify the nature of his structuralist views, and (ii) to show that they were not casual, or tangential to the rest of his philosophy. His views about structure are entwined with several themes in his philosophy of mathematics. We will begin with some general structuralist sympathies, which emerge from his reflections on mathematical understanding. We will then work toward more specific, and stronger, philosophical views about the nature of mathematical structures, which appear in his thoughts about geometry and group theory. Like Shapiro, Poincaré contrasts two main philosophical views about the group structure in terms of whether or not it should be considered as prior to, and independent of, its relevant mathematical systems. Also with Shapiro, and somewhat surprisingly, Poincaré explicitly endorses the priority view about the group structure.

2.2.1. Remarks on Mathematical Understanding

Poincaré famously comments on mathematical understanding and insight, referring to phenomena such as "seeing the whole" and the view "from afar." Some of these are vague, negative remarks against the role of logic in mathematics (sometimes against logicism more specifically), while others seem more positive, as genuine attempts to articulate the nature of mathematical understanding. That logical reasoning alone does not constitute understanding seems obvious. The hard task is to say what more is needed.

Starting with the negative, Poincaré complains that "the logician cuts up, so to speak, each demonstration into a very great number of elementary operations." But as we all know, following individual inference steps does not amount to understanding even a straightforward proof: "we shall not yet possess the entire reality; that I know not what, which makes the unity of the demonstration" (1900, V, 1017). His point is that understanding a proof is something over and above understanding the individual inferences. Obviously, making individual proof-inferences will not suffice for creating new mathematics; here Poincaré asserts that the same holds even for understanding an existing proof.[8]

[8] This view may call to mind Wittgenstein's remarks about proofs needing to be surveyable (see Wittgenstein 1989). However, Poincaré does not propose surveyability as a requirement for *proofs*; he connects it only to *understanding* proofs. (Of course, interpreting Wittgenstein on this and similar issues is not simple.)

He provides a famous analogy to make this point about the holistic nature of mathematical understanding: "A naturalist who never had studied the elephant except in the microscope, would he think he knew the animal adequately?" ([1908] 1982, Book 2, chap. 2, sec. 6, 436). In addition to making a part-whole contrast, Poincaré is alluding to a "big picture" or the "forest for the trees" idea. Analyzing elephant cells does not provide understanding of the animal as a whole, an understanding that can be gained only by observing the living animal. Similarly, it is not that Poincaré saw no value in attending to local logical inferences; rather, his point is that this type of focus does not *suffice* for—or constitute the whole of—understanding a proof, or a mathematical fact. "This view of the aggregate is necessary for the inventor; it is equally necessary for whoever wishes really to comprehend the inventor. Can logic give it to us? No" (1900, V, 1018).

What can give a view of the whole, if not logic? That is much harder to articulate. In these and similar passages Poincaré sets up a contrast between *rigor* and *understanding* in mathematics. "Rigor" here means focusing on the "parts"—the formal, symbolic definitions, explicit deductive inferences, etc. "Understanding," in contrast, is presented as something that involves the "whole," something that transcends rigor concerning the parts. This includes grasping the unity of a proof (1900, V), the historical origins of precise definitions (1900, IV; [1908] 1982, Book 2, chap. 2), the point of a mathematical question (1900, IV), and the ability to invent (1900, V). The claim is that to understand and create new mathematics, one needs this perspective of the whole.

At times Poincaré mentions intuition in this context. "We need a faculty which makes us see the end from afar, and intuition is this faculty" (1900, V, 1018). The appeal to intuition here may seem psychologistic, and certainly it is distinct from his semi-Kantian appeal to mathematical intuition (which will be addressed later). Though Poincaré's remarks are vague, both the metaphors and the reference to intuition point to the idea of transcending individual results and local logical inferences. How is this related to structuralism?

Consider, for example, the metaphor of "seeing from afar"; plausibly this includes the ability to connect distinct results and even different areas of mathematics. To do so requires a more abstract, higher-level, perspective—a perspective that seems generally structural. At the very least, structure is something that different areas of mathematics *can* have in common. For example, as we will later see, Poincaré cites the group structure as what is common to various mathematical systems. And in a chapter on the relation between mathematics and physics he writes rather poetically:

> What has taught us to know the true, profound analogies, those that the eyes do not see but reason divines?

> It is the mathematical spirit, which disdains matter to cling to pure form. This it is which has taught us to give the same name to things differing only in material, to call by the same name, for instance, the multiplication of quaternions and that of whole numbers. . . . He sees best who stands highest. ([1905] 1958, chap. 5, II, 77–78)

The view from afar, or above, is where one can "see" structural, relational similarities between systems "differing only in material."[9]

Structure also underpins the perception of beauty according to Poincaré, which, in turn, supports understanding. When we perceive the beauty of a piece of mathematics, he thinks, we understand it better, and vice versa. Further, he argues that mathematical beauty involves the more primitive properties of order and unity. Good ideas, the impression of elegance, the use of analogy, and the importance of generality all depend on perceptions of order and unity, in his view. Successful creative work, he argues, is guided (consciously or unconsciously) by "the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance" ([1908] 1982, Book 1, chap. 3, 391). For him, these "feelings" are all grounded in unity and order, which is both aesthetically pleasing and useful in "guiding" the mind to fruitful results. This view—that perceptions of order and unity facilitate understanding—fits well with basic structuralism, since structure is an organizing tool. Poincaré's conception of mathematical understanding is thus harmonious with the view that mathematics is (at least largely) about abstract structure.[10]

2.2.2. The Subject Matter of Mathematics

In addition to the epistemic view that the perception of structure facilitates mathematical understanding, Poincaré also expresses structuralist views about the subject matter of mathematics. In fact these claims seem parallel. Just as the general "overview" perspective is essential for mathematical understanding, so a "bigger picture" perspective is critical for the subject matter, since the significant results are general and have broad scope. In addition, as we saw earlier, Poincaré makes the standard structuralist point that mathematics is "about" relations rather than objects. His emphasis on both general truths over particular truths, and relations over particular objects, reflects a structuralist vision of the subject matter and methodology of mathematics.

[9] One may perceive here an early expression of something like the category-theoretic perspective. For more on the path toward category theory, see the chapters on Noether (by Audrey Yap), Bourbaki (by Gerhard Heinzmann and Jean Petitot), and Mac Lane (by Colin McLarty) in this volume.

[10] I attempt to more fully address this connection between mathematical structure and understanding in Folina 2018.

Regarding the importance of generality, he remarks: "So a chess player, for example, does not create a science in winning a game. There is no science apart from the general" (1894, II, 975). Further, he aligns generality in mathematics with infinity, claiming that without the idea of mathematical infinity "there would be no [mathematical] science, because there would be nothing general" (1894, V, 979).

About relations, he famously says of Dedekind cuts:[11]

> Mathematicians study not objects, but relations between objects; the replacement of these objects by others is therefore indifferent to them, provided the relations do not change. The matter is for them unimportant, the form alone interests them.
>
> Without recalling this, it would scarcely be comprehensible that Dedekind should designate by the name incommensurable number a mere symbol, that is to say, something very different from the ordinary idea of quantity, which should be measurable and almost tangible. ([1902] 1952, chap. 2, 44–45)

The first paragraph seems a canonical statement of methodological structuralism. However, what about the negative tone of the second paragraph?

The context is relevant. In this section of the chapter Poincaré not only explicates, he also criticizes, work he considers reductionistic.[12] For example, he is unhappy about attempts to define the continuum "without using any material other than the whole number" (44). He objects as follows after explaining Dedekind cuts.

> But to be content with this would be to forget too far the origin of these symbols; it remains to explain how we have been led to attribute to them a sort of concrete existence, and, besides, does not the difficulty begin even for the fractional numbers themselves? Should we have the notion of these numbers if we had not previously known a matter that we conceive as infinitely divisible, that is to say, a continuum? (45–46)

Poincaré is thus not just *citing* Dedekind cuts as an example of the fact that mathematics is about abstract structure; he is *criticizing* Dedekind cuts as a theory of the real numbers. He seems to regard it as too abstract and too formal, or

[11] For Dedekind, including a further discussion of this remark by Poincaré about his use of cuts, see again the chapter by Ferreirós and Reck in this volume.

[12] He cites the "Berlin school" here, and "Kronecker in particular," for these sins, but then goes on to discuss Dedekind in some detail (who is usually not considered as having belonged to the Berlin school). His point seems to have been against reductionist programs generally without differentiating between the various motives and origins of specific projects.

symbolic.[13] At least, it is insufficient if a theory should provide understanding. Since structuralism is associated with the increasing abstraction and formalization of the content of mathematics, this critique of Dedekind (and similar projects) may appear to weaken the structuralist interpretation of Poincaré. Let me flesh out this concern before attempting to assuage it.

In contrast with a formal/symbolic theory of the real numbers, as one might regard Dedekind cuts, Poincaré insists that understanding the mathematical continuum comes from our relating it to experience. He makes the following familiar argument: we experience a physical continuum, but this leads to contradictions owing to our limited senses. That is, we can experience three lengths or weights as A = B and B = C. But we can also tell that A is longer or heavier than C; so A > C; but now this is inconsistent. We solve this by citing the limited nature of our sense perceptions; so we suppose that even though A seemed the same weight as B and B seemed the same weight as C, at least one of these measurements was not quite right. That is, we *conceive* the things measured in terms of quantities that are further divisible—beyond our capacities to sensibly distinguish such differences. In addition, once we interpolate between two given quantities, "we feel that this operation can be continued beyond all limit" ([1902] 1952, chap. 2, 48). We thus suppose the operations are indefinitely iterable, which leads to the conception of everywhere dense sets like the rationals. Irrationals are then postulated owing to theoretical gaps in the rationals. Poincaré states the point thus: "the mind has the faculty of creating symbols. . . . Its power is limited only by the necessity of avoiding all contradiction; but the mind only makes use of this faculty if experience furnishes it a stimulus thereto" (49). Returning to our issue, the problem with Dedekind cuts is that the theory can be presented completely in its abstract, formal guise; and this would make the real number system seem independent of both geometry and experience. But analysis emerges from the union of geometry with the needs of physics and arithmetic, which the subject matter should reflect.[14]

In fact, Poincaré was generally suspicious of mathematics that is detached from history and experience, and when applied to this case, this may seem contrary to structuralism or the structuralist enterprise.[15] For example, regarding nowhere differentiable continuous functions he says: "Instead of seeking to reconcile analysis with intuition, we have been content to sacrifice one of the two, and as analysis must remain impeccable, we have decided against intuition"

[13] Poincaré *was* anti-logicist, but his concern here seems more general. He is questioning our ability to understand formal, symbolic definitions without supplementary information, whether or not they are part of a logicist program.

[14] His critique implies that he thought the connections to experience must go beyond merely motivating or teaching the theory.

[15] Of course in another sense, attention to the "larger" view is an expression of structuralism—as just argued.

([1902] 1952, chap. 2, 52). The remark is clearly a complaint or a regret; it is not merely a *report* of mathematical progress by increasing formalization. I think it is fair to say that Poincaré was a bit ambivalent about some of the changes in mathematics associated with the development of the structuralist viewpoint.

We can now return to our concern, which is that Poincaré's canonical statement of methodological structuralism is accompanied by critical remarks about the central example of Dedekind cuts.[16,17] More evidence and more particulars regarding the nature of his structuralist commitments will clarify and improve our case that Poincaré genuinely embraced structuralism. Geometry provides this support.

2.2.3. Geometric Conventionalism

Poincaré's conception of the subject matter of geometry provides some key, added evidence of his structuralism. Now, Shapiro connects both axiomatics and the idea of implicit definition with structuralism, singling out Hilbert's 1899 *Grundlagen der Geometrie* as "the culmination of a trend toward structuralism within mathematics" (sec. 1).[18] Why is the axiomatic method associated with structuralism? Because along with the idea of implicit definition, the axiomatic method *lifts* the subject matter of mathematics up to a higher, more abstract, level. For example, rather than thinking of geometry as having a single, object-level subject matter, which the axioms are *about*, the "axiomatic view" is that axioms are about *whatever* systems fulfill the criteria they jointly stipulate. So the axiomatic method changes our perspective from *single* subject matter to a *multiplicity*, or set, of possible interpretations. Indeed, the ascendance, or abstraction, of subject matter was important for 19th-century developments in both geometry and algebra.[19]

Poincaré clearly conceives the subject matter of geometry similarly to Hilbert—in this "elevated" way. He is famous for claiming that geometric axioms are implicit definitions, or conventions:

[16] To elaborate a bit on Poincaré's mixed feelings in these passages: he recognized that the more formal, abstract "structuralist" perspective enables advances in mathematics, so it is crucial. But he also recognized that abstraction makes even ordinary mathematics harder to understand. Thus, more formal, symbolic methods are desired at times; but these must be supplemented to facilitate "understanding."

[17] One might respond here by pointing out that these criticisms of Dedekind are strictly *epistemic*; they do not undermine the general structuralist view regarding the *subject matter* of mathematics. Nevertheless, more evidence will solidify my interpretation. Additionally, the epistemology of mathematics should cohere with its subject matter.

[18] I assume that "structuralism within mathematics" is (essentially) what I (after Erich Reck and others) have been calling "methodological structuralism."

[19] This is of course historically complex and interesting, though I cannot here address it further.

> The axioms of geometry, therefore, are neither synthetic *a priori* judgments nor experimental facts.
>
> They are conventions; our choice among all possible conventions is guided by experimental facts; but it remains free and is limited only by the necessity of avoiding all contradiction. . . .
>
> In other words, the axioms of geometry (I do not speak of those of arithmetic) are merely disguised definitions.
>
> Then what are we to think of that question: Is the Euclidean geometry true?
>
> It has no meaning.
>
> As well ask whether the metric system is true and the old measures false. ([1902] 1952, chap. 3, 65)

That geometry is based more on choice than truth expresses the fact that like Hilbert, Poincaré viewed geometric axioms as "disguised" or implicit definitions. The comparison to measurement systems—for which the main criteria for acceptability are consistency and convenience rather than truth—shows that Poincaré sees geometry as at least partly detached from a truth-determining subject matter.

It may be worth elaborating a bit on the differences between geometry and arithmetic, as Poincaré saw it. To him, arithmetic has an intuitively grounded subject matter. In contrast, intuition does not anchor geometry in a similar way.[20] We have no direct intuition of points: "What is a point of space? Everybody thinks he knows, but that is an illusion" ([1902] 1952, chap. 5, 89–90). Nor does intuition decide what is straight: "I grant, indeed, that I have the intuitive idea of the side of the Euclidean triangle, but I have equally the intuitive idea of the side of the non-Euclidean triangle. Why should I have the right to apply the name of straight to the first of these ideas and not to the second?" ([1905] 1958, chap. 3, I, 37–38). This is why conventional choices, or implicit definitions, are necessary. We decide which axiom system is most convenient, and this decision determines what lines will be considered straight when using that system; that is, the axiom system is the implicit definition of "straight line." In this way, geometry is no longer seen as *reflecting* a single definite subject matter (though it was once so regarded).

What we might call "mathematical geometry" thus occupies a more abstract, structural, position than ordinary Euclidean geometry. In contrast, ordinary geometry—working within a particular geometric system—is in a sense closer

[20] This is not to say intuition does not anchor geometry at all. Indeed geometry is supported both by the intuitive continuum and by the intuition of indefinite iteration. (He even refers to "geometric intuition" in later work (e.g., [1913] 1963, 26–27and 42–44). The difference is that, unlike arithmetic, intuition does not yield any particular geometric system as *true*.

to applied mathematics, since it lies at a lower level of abstraction. In any case, like Hilbert, Poincaré's conception of geometric axioms as implicit definitions provides further evidence of his structuralist perspective, at least regarding geometry.

Before moving on, I'll note that in addition to his view of axioms as implicit definitions, Poincaré also calls certain key concepts "implicit axioms." Here he draws attention to the existence of concepts or principles that unite different systems. While changing an axiom creates a different axiom system, Poincaré's point is that despite the differences, there are often important connections, or relations, between the systems ([1902] 1952, chap. 3, 60–62). For example, rigid body motion is presupposed by several geometric systems; but its possibility is neither self-evident nor analytically true. So Poincaré considers rigid body motion to be an "implicit axiom," in that it acts as a unifying principle for the geometries of constant curvature. As we shall see, the group concept plays a similar role. My point is that it is not only the *differences* between systems—e.g., the proliferation of geometries—that can be linked to the structuralist perspective. Emphasizing the *links* between the different systems—the unifying concepts and principles—also expresses structuralism. Indeed, in my view, Poincaré's emphasis on unifying concepts provides an even stronger connection to structuralism than his view of axiom systems as implicit definitions. Let us now turn to a key example—that of the unifying concept of group.

2.2.4. The Group Concept

In addition to conceiving the geometric axioms as implicit definitions, Poincaré further emphasizes geometric *form*, and the group concept is central here. An interesting twist is that the group concept is a priori for Poincaré—not as an intuition, or form of sensibility, but as a "form of our understanding" ([1902] 1952, chap. 4, 79). It underpins our ability to conceive geometry from the more abstract perspective, which, in turn, helps us make sense of multiple possible geometric systems.

> What we call geometry is nothing but the study of formal properties of a certain continuous group; so that we may say, space is a group. The notion of this continuous group exists in our mind prior to all experience; but the assertion is no less true of the notion of many other continuous groups; for example, that which corresponds to the geometry of Lobatchevski. (1898, Conclusions, 1010)

A variety of continuous groups are a priori possible, and are studied in mathematics. The choice for a theory of physical space (Euclidean or non-Euclidean, 3-dimensional or 4-dimensional) is conventional, depending on experience, science, and other factors.[21] Because the mathematics behind geometry is common to a variety of options, it—*mathematical* geometry—lies at a more abstract level than work within particular geometric systems. The group concept, and the idea that the various geometries are simply different continuous groups, facilitates the "ascendance" to this more abstract mathematical perspective. So Poincaré's appeal to the group concept, and its role in articulating the abstract perspective of mathematical geometry, provides even more evidence of his methodological structuralism.

Crucially, however, Poincaré's view about groups goes further. It not only furnishes a clear statement of basic structuralism; it also includes properly philosophical views about the metaphysical nature of groups as well as our knowledge of them. Thus, his view here clearly advances beyond methodological structuralism to a more philosophical position about (at least some) mathematical structure.

One addition is an epistemological claim, noted previously. The apriority of the group concept, and the view that this a priori status provides mathematics with a unifying ideal, transcends basic methodological structuralism (which mainly concerns the general subject matter of mathematics). For Poincaré, the group concept provides a perspective from which to consider, compare, and unify the different geometries (as well as other structures). His views about the group concept thus address the epistemology of geometry, in its new, more abstract, guise.

But he also makes an ontological claim. That is, in addition to the apriority, and unifying role, of the group concept, Poincaré adds that the group structure is *prior to* the systems falling under it. The following remark, in particular, shows him asserting a view similar to Shapiro's "*ante rem*" structuralist (in the "structure first" sense):

> We must distinguish in a group the form and the matter. For Helmholtz and Lie the matter of the group existed previously to the form. . . . The number of dimensions is therefore prior to the group. For me, on the contrary, the form exists before the matter. The different ways in which a cube can be superposed upon itself, and the different ways in which the roots of a certain equation may be interchanged, constitute two isomorphic groups. They differ in matter only. The mathematician should regard this difference as superficial, and he should no more distinguish between these two groups than he should between a cube

[21] There is a large literature on this; for example see the anthology de Paz and DiSalle (2014).

> of glass and a cube of metal. In this view the group exists prior to the number of dimensions. (1898, Form and Matter, 1009–1010)

That the important mathematical properties of geometry concern form rather than matter is simply methodological structuralism. But that the "form exists before the matter" is a much stronger view, one that coincides with the "structure first" view of *ante rem* structuralism. For form to exist *before* matter—for it to be prior— it must also be *independent* of matter or specific systems.

For Poincaré, the group structure, or form, is relatively independent—independent of the mathematical objects or systems exemplifying it. Now, Poincaré was not a realist about mathematical existence, so his view is not that of (Shapiro's) Platonic *ante rem* structuralism. Yet his view perfectly matches the independence and priority aspects of the *ante rem* view. Because the group concept is a priori, the group structure is epistemically prior to any particular group. And because, as he asserts, the form of a group exists prior to any specific group, the group structure is prior in (some sense of) existence as well. Detached from Platonism, the category of *ante rem* structuralism simply indicates the *relative* independence and priority of structures to their systems and objects. As we just saw, this is precisely what Poincaré asserts about the group structure.[22]

3. Structuralism and Other Issues in Poincaré's Philosophy

Supposing that Poincaré's view of groups matches that of *ante rem* structuralism, how does this fit with his other philosophical commitments? Are there any other structures that come "first," or are groups unique on this matter? Was Poincaré consistently anti-realist in his philosophy? If so, how does his mathematical constructivism, or anti-realism, combine with this view about groups; that is, can one be a constructivist and still think that any mathematical structures are prior to and independent of their instances?

I will now take up these last three questions, each in its own section. Starting with the nature and extent of Poincaré's constructivism, I will present his semi-Kantian conception of mathematical intuition as structuralist. That is, I'll argue that intuition for Poincaré regards mathematical structures, and moreover, that the intuitive structures come "first" in a way similar to that of the group structure. Intuition on my reading thus adds to the stock of abstract structures that come "first." The second question is whether or not Poincaré is consistently anti-realist. Here I note that his philosophy of natural science is generally considered

[22] Again, I will come back to this issue subsequently.

a form of structural realism. I argue, however, that his views about mathematical structures and those about natural structures are independent of one another. So his realism about some scientific structures is consistent with his general constructivist, or anti-realist, position on mathematical existence.[23] Last, I address the apparent tension between his anti-realism about mathematics, on the one hand, and the view that mathematical structures can come "first," on the other; that is, I try to make sense of how one can be a "constructivist *ante rem* structuralist."

3.1. Intuition and Structuralism

It may be easy to understand how group theory fits with structuralism, but intuition may seem more puzzling. As for Kant, Poincaré sees intuition as necessary for both the subject matter of mathematics and our knowledge of it. Yet, as argued earlier, he also endorses a structuralist view of the subject matter of mathematics. If intuition governs the content and our knowledge of mathematics, and mathematics is about structure, then intuition must provide insight into structure. Intuition for Poincaré delivers an epistemology of mathematics that complements the new structuralist conception of its subject matter.

Poincaré is clearly a "constructivist" given his repeated claims to support a semi-Kantian conception of mathematics, including mathematical intuition. But his view is distinctive. For Kant, intuition in mathematics is spatiotemporality, the a priori form of all experience; there is, for Kant, no specifically *mathematical* intuition. Furthermore, the role of intuition in mathematics is quite complicated, having to do with the necessity of input from space and time to instantiate the mathematical concepts via a process he calls "construction of concepts."

Like Kant, Poincaré defends two a priori intuitions in relation to mathematics; but instead of space and time, he cites the intuition of the continuum and the intuition of indefinite iteration. These are more abstract and closer to mathematical intuitions. In some ways they are like "stripped down" versions of space and time: spatiotemporality minus most of the sensorial aspects we associate with it. To put it another way, intuition for Poincaré is more cognitive and less connected to ordinary sense experience than space and time are for Kant. Also in contrast with Kant, there is no explicit reliance on "construction of concepts" in Poincaré.[24]

[23] More than consistent, I actually find the two views mutually supporting despite the appearance of a contrast between them.

[24] For Kant constructing concepts is distinctive of mathematical methodology, and means something quite specific having to do with considering/exhibiting arbitrary instances of mathematical

3.1.1. Arithmetic and Intuition

In arguments against programs like logicism, Poincaré emphasizes both the centrality of the principle of induction and its basis in intuition. As mentioned, unlike Kant, Poincaré appeals to indefinite iteration rather than time. But as in Kant, intuition's role in grounding mathematical knowledge makes that knowledge synthetic a priori. So the dependence of induction on intuition makes our knowledge of induction, as well as any knowledge it yields, synthetic a priori for Poincaré.

This is clear in his early writings on the nature of mathematical reasoning, where he argues that the power of mathematical reasoning—its ability to transcend the merely tautological—springs from the principle of induction. And since induction is grounded in intuition, it's really intuition that gives mathematical reasoning this power.

> Why then does this judgement force itself upon us with an irresistible evidence? It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience can only give occasion for using it and thereby becoming conscious of it. (1894, VI, 979–980)

The intuition of indefinite iteration is a mental capacity that allows us to conceive of certain sets, or certain sequences, by conceiving of the way they can be produced by us: iteratively, step by step. Poincaré's skepticism about transfinite cardinals, as well as any philosophy that accepts actual infinity, is related to this idea that sets must be conceived (if not strictly constructed) by envisioning our producing, or "running through," their elements in a stepwise fashion.

Though essential for conceiving discrete infinite sets, Poincaré's main explanation of the importance of indefinite iteration focuses on how we understand induction rather than how we construct the mathematical sets that induction might target. Further, his explanation of why intuition is required for induction articulates the intuition as insight into a certain type of mental process. It is the focus on *type* in these and surrounding arguments that supports the connection to structuralism. Poincaré claims the same intuition is the basis for inferences about very different sorts of objects. What unifies these, what the different systems of objects have in common, is their structure, or order type, not their

concepts in space and/or time. Mathematical "constructions" are central to Poincaré's view, but their role is less specific; sometimes "construction" simply means defining and perhaps comprehending. (See later in this chapter for one such use regarding the continuum.)

content.[25] Thus the main intuitive basis for arithmetic is that which provides cognition of structure. Let me explain this in a bit more detail.

For Poincaré, the paradigm case of knowledge that requires the intuition of iteration is inductive knowledge. "The essential characteristic of reasoning by recurrence [induction] is that it contains, condensed, so to speak, in a single formula, an infinity of syllogisms" (1894, V, 978). The puzzle is this: if induction implicitly contains an infinity of inferences, how can we recognize it as yielding truth? The answer for Poincaré is intuition. We can arrange some initial inferences as the following: property P is true of 0, and if true of n then true of n + 1, so property P is true of 1. Since P is true of 1, and if true of n then true of n + 1, the property P is also true of 2. And so on. Poincaré notes that when the inferences are ordered this way, they are "arranged in 'cascade' "; because of this arrangement we can see that they will continue to be valid indefinitely, or infinitely. And intuition, the mind's ability to conceive "the indefinite repetition of the same act when once this act is possible" (1894, VI, 979), is what provides the necessary insight into the infinite chain of modus ponens steps constituting the "cascade."

Not only do we see induction as thus leading to truth, we recognize it as necessary. Unlike empirical induction, "Mathematical induction, that is, demonstration by recurrence, on the contrary, imposes itself necessarily because it is only an affirmation of a property of the mind itself" (1894, VI, 980). Induction affirms the property of the mind associated with the intuition of iteration. Thus, the paradigm case of intuitive knowledge is not (acquaintance) knowledge *of* a series of objects, such as the natural numbers, but (propositional) knowledge *that* a type of inference preserves truth. The intuitive basis for arithmetic is thus more distant from sensibility than an account focused on the potential construction of finite objects (such as sequences of strokes for cardinal numbers). In short, it is more epistemic, more abstract, and less ontological.

Moreover, Poincaré defends his semi-Kantianism about arithmetic against challenges from logicism and formalism/axiomatics by arguing that different types of induction are really the same principle, all depending on the same intuition.[26] The kernel of the argument is expressed at least by 1894: "If we look closely, at every step we meet again this mode of reasoning, either in the simple form we have just given it, or under a form more or less modified" (1894, IV, 978). He also claims at this time that induction cannot be demonstrated in a non-circular way (1894, VI, 979), though he doesn't provide an argument for this until a later series of circularity objections to logicism (Poincaré [1905] 1996, [1906a] 1996, and [1906b] 1996).

25 Their order type being that of a simply infinite system.
26 That is, induction, or really iteration, acts as another unifying principle for Poincaré.

I cannot go into the details of his circularity arguments here.[27] Though his point at first regards the different ways we reason inductively about numbers, he later adds that *metatheoretic* uses of induction depend on the same underlying intuition of iteration.[28] If metatheoretic uses of induction and induction on numbers depend on the *same* intuition, intuition is formal, and independent of any particular content.

For example, in discussing any proof that the arithmetic axioms are consistent, he writes, "recourse must be had to procedures where in general it is necessary to invoke just this principle of complete induction which is precisely the thing to be proved" ([1905] 1996, IV, 1027). So a consistency proof *about* arithmetic uses "precisely" the same principle as when reasoning inductively *in* arithmetic. Since the two uses obviously will concern different *objects*, any precise sameness must regard form or type. His associating intuition so closely with a form of reasoning applicable to various domains, rather than (just) a method for constructing objects to constitute a domain, gives this intuition a more abstract feel. That is, similarly to how the group concept facilitates the ascendance to a more abstract way of thinking about geometry, intuition is appealed to here to explain a more abstract way of thinking about inductive reasoning and the domains to which it applies. In short, what grounds and guides induction is intuition of structure.

Indeed, Poincaré explains why this form of reasoning can be used in such different contexts by invoking structuralist terms—by reference to the common underlying structure as an "ordinal type."

> Thus one envisages a series of reasonings succeeding one another, and one applies to this succession, regarded as an ordinal type, a principle that is true for certain ordinal types, called finite ordinal numbers, and which is true for these types precisely because these types are by definition those for which it is true. ([1906a] 1996, XXIII, 1043)

Though the remark is a little cryptic, I take it as supporting a strong connection between intuition and structure. Whether it is induction about numbers, or about sequences of inferences, the reasoning depends on the fact that intuition enables both the understanding of induction and the cognition of the simply infinite systems to which induction can be applied. Without this intuition, this insight, we (finite thinking beings) would not be able to do mathematics about discrete infinite systems, and we would not be able to see that the principle of induction is true.

27 But see Folina (2006) and Goldfarb (1985) for opposing views on these arguments.
28 This is how he justifies the earlier circularity claim.

On my interpretation, then, Poincaré thought of the natural number *structure* as—like the basic group structure—more fundamental than any particular system of natural numbers. His stress on the equivalence of the various uses of induction ([1906a] 1996, 1050) supports this interpretation, as does his description of intuition as insight into an abstract, or type of, mental capacity. Intuition for Poincaré is a *format*—for producing, understanding, and reasoning about various systems of objects. Since it is what enables us to produce infinite sets (insofar as we can—in our conceiving them) it is prior to those sets. Thus, like the group concept, the natural number structure, as given by a priori arithmetic intuition, is another structure that comes "first."

3.1.2. Geometry, Analysis, and Intuition

Despite his famous conventionalism about geometry (more specifically, the choice of a geometry for physics), Poincaré also endorses "geometric intuition" ([1913] 1963, 43–44). But "geometric" is a bit of a misnomer here. Geometric intuition does not yield a particular set of geometric truths, or knowledge that a particular geometric structure is true. So it is not an intuition of geometry. Rather, it provides more general cognitive access to physical and mathematical continua, via the "intuitive notion of the continuum."

> I shall conclude that there is in all of us an intuitive notion of the continuum of any number of dimensions whatever because we possess the capacity to construct a physical and mathematical continuum; and that this capacity exists in us before any experience, because, without it, experience properly speaking would be impossible and would be reduced to brute sensations. . . . And yet this capacity could be used in different ways; it could enable us to construct a space of four just as well as a space of three dimensions. ([1913] 1963, 44)

Like indefinite iteration, the intuition that lies behind the more "spatial" areas of mathematics, such as geometry, analysis, and topology, is a mental capacity—a capacity for constructing various types of abstract spaces.[29]

Poincaré associates this intuition most closely with "analysis situs," or topology, "the true domain of geometric intuition" ([1913] 1963, 42). But it also supports other areas of mathematics that are "spatial," including geometry and analysis. For example, in assessing Hilbert's axiomatic approach to geometry, Poincaré argues that the axioms of order are genuine intuitive truths ([1913] 1963, 43). They are central to topology, and they also play a fundamental role in our cognition of ordinary (metric) geometry.

[29] Though in rather Kantian style he also asserts that it is a form of experience—necessary for experience as we know it.

Because the intuitive continuum functions as a template for cognizing and defining various mathematical and physical continua, it is structural in a similar way to iteration. For example, it too *precedes* any particular instantiation, since the intuition is what "enable[s] us to construct" the continuous space we wish to consider. Thus, the intuitive continuum joins arithmetic intuition and the group concept to provide a third example of a mathematical structure that comes "first," i.e., before its instances.

To conclude this section, I aimed to show that Poincaré's conception of mathematical intuition harmonizes with the structuralist interpretation. Indeed, it strengthens it by adding new mathematical structures that come "first": the simply infinite structure and the (n-dimensional) continuum. Like the group concept, intuition precedes its uses. Recall that the group concept is a priori and the group structure is prior to its instances. Similarly, mathematical intuition is a priori; it enables synthetic a priori mathematical knowledge about infinite domains by providing insight into infinite structures, which are prior to their instances. The structures supplied by intuition are prior for Poincaré because we need the intuitive structures in order to "construct" (or conceive) the mathematical domains for which they provide the template. Thus, Poincaré's conception of intuition strengthens and adds to the "structure first," *ante rem*, interpretation.

3.2. Structural Realism and Mathematical Structuralism

Before turning to the question of *how* Poincaré (or anyone) could combine a semi-Kantian anti-realist view of mathematics with a "structure first" view, we will first briefly consider the fact that he is also commonly associated with realism—structural realism in the philosophy of (natural) science. This is roughly the view that although science does not generally provide us with absolute truths about objects in nature, it can yield knowledge of structures in nature. Structural realists acknowledge the so-called pessimistic meta-induction—scientific theories come and go—and they concede from this that it is naive to think that any one scientific theory provides eternal knowledge or insight into the essences of things. But this does not mean that science provides no knowledge at all. Instead, they maintain, there is evidence that science yields knowledge of the structure of reality. The success of science and the persistence of form through theory change are two supporting arguments commonly deployed by structural realists.

Poincaré has been cited as one of the first to articulate structural realism.[30] As I see it, the structuralist perspective guided his work in both mathematics

[30] See the classic piece by Worrall (1989); but also see Brading and Crull (2017) for a more modest, middle position on the "realism" in Poincaré's structural realism.

and physics, providing a cornerstone for his overall scientific epistemology. Structural realism about science and structuralism about mathematics are thus in a sense two sides of one epistemic coin.

One can find at least four arguments from Poincaré supporting structural realism. There are two familiar "negative" arguments against naive realism: (i) the privacy of acquaintance knowledge and the resulting weakness of direct realism,[31] and (ii) the acknowledgment of scientific change—the so-called pessimistic meta-induction ([1905] 1958, chap. 6). Poincaré also provides two common "positive" arguments aiming to rescue scientific knowledge from a more skeptical viewpoint, to which the two negative arguments might seem to lead. These are claims about (iii) the success of science ([1902] 1952, Introduction, 28) and (iv) the persistence of form through theory change ([1902] 1952, chap. 10, 153). His "rescue" leads to a type of structural realism.

Poincaré agrees that science is neither a direct reflection of reality nor is it simply cumulative: revisions and revolutions are part of science. Yet he resists skepticism. That is, despite the fact that scientific theories often change (the basis for the pessimistic meta-induction) Poincaré was optimistic about scientific knowledge. For example, though he emphasizes scientific conventions, it is a mistake to think this is the view that science is *just*, or *mainly*, based on decisions. That is, overemphasizing the freedom of conventions makes science seem arbitrary.

> If this were so [if science were arbitrary], science would be powerless. Now every day we see it work under our very eyes. That could not be if it taught us nothing of reality. Still, the things themselves are not what it can reach, as the naïve dogmatists think, but only the relations between things. Outside of these relations there is no knowable reality. ([1902] 1952, Introduction, 28)

There may be other realms of truth, or other ways of knowing reality; certainly the reality we can know is limited. Direct acquaintance knowledge is not objective for it is not even intersubjective; and things in themselves cannot be known at all.[32] But from the success of science Poincaré concludes that we do have objective knowledge—that of general, relational facts.

[31] "The sensations of others will be for us a world eternally closed. We have no means of verifying that the sensation I call red is the same as that which my neighbor calls red. . . . In compensation, what we shall be able to ascertain is that, for him as for me, the cherry and the red poppy produce the same sensation. . . . The relations between the sensations can alone have an objective value" ([1905] 1958, chap. 11, 136).

[32] Along these lines, just as logical positivism can be seen as an adjustment of Kant's vision, one can see Poincaré's structural realism similarly.

In addition, there is evidence that science uncovers the structure of reality in particular: this is the persistence of equations, or equation-forms, across theory change.

> Not only do we discover new phenomena, but in those we thought we knew, unforeseen aspects reveal themselves. . . . Nevertheless the frames are not broken. . . . Our equations become, it is true, more and more complicated, in order to embrace the complexity of nature; but nothing is changed in the relations which permit the deducing of these equations one from the other. In a word, the form of these equations has persisted. ([1902] 1952, chap. 10, 153)

Putting some of this together, we can fill in our picture a bit. Essences of things, or things in themselves, are not knowable. This resonates with his broadly Kantian epistemic vision, and it explains why the "images" of things shift with changes in scientific theory. But skepticism on these grounds is "superficial" according to Poincaré ([1902] 1952, chap. 10, p. 140). Instead, he believes that science reveals relations that actually exist in nature: "equations express relations, and if the equations remain true it is because these relations preserve their reality" ([1902] 1952, 140). Together, these views express a fairly straightforward version of scientific structural realism.[33]

Now, structural realism about science is neither necessary nor sufficient for structuralism about mathematics. Yes, mathematics is the "language" of science; but that scientific theory reveals the structures, or relations, of *nature* is simply different from the view that the subject matter of *mathematics* is abstract structure. Of course, the two views are not completely independent, or merely consistent. Though Poincaré was realist about (some) scientific structure and anti-realist about mathematics, the emphasis on structure in both views makes them harmonious and mutually supportive. In particular, they share a compelling view about perspective. The perspective of object-level content is "superficial" both in

[33] Putting "Kantian" in the same paragraph with "realism" may jar some readers. However, I do think Poincaré held versions of both views. With Kant we cannot know the things in themselves; also with Kant, mathematics provides a synthetic a priori foundation for scientific knowledge. Unlike Kant, Poincaré expresses confidence that the persistent structures and relations revealed by scientific inquiry reflect the way things are in nature, rather than just the way we are constituted to experience and/or conceptualize nature. There is a hint of Darwinism in this view, reminiscent also of Hume's "pre-established harmony" between nature and ideas (Hume, *Enquiry* Part V, last two paragraphs). For Poincaré, general, simple laws are most interesting and most beautiful—perhaps because we are constituted to appreciate them; but they are also necessary for science. If there were no general laws in reality, if there were 60 million chemical elements or only individuals but no biological species, "In such a world there would be no science; perhaps thought and even life would be impossible, since evolution could not there develop the preservational instincts" ([1905] 1958, Preface, 5; see also chap. 10, sec. 3, 115–122, for the view that there are prescientific "crude" facts, which science merely "translates" rather than "creates"). Though his arguments focus mostly on epistemological issues, his conclusions clearly endorse realism about at least some structural facts.

natural science (e.g., [1902] 1952, 140) and in mathematics (e.g., 1898, Form and Matter, p. 1009). However, this epistemic point—that the higher-level, structural perspective is crucial in both natural science and mathematics—is indifferent to the metaphysical question of whether or not the relevant structures exist independently of the scientists and mathematicians.

3.3 Constructivism versus "Structure First"?

In section 1, I argued that within Shapiro's basic taxonomy, the category of *ante rem* structuralism best fits Poincaré's further philosophical assertions about the group structure. And in section 3.1, I argued that we can extend the *ante rem* interpretation to the structures given by mathematical intuition. Indeed, as I reconstruct Poincaré's vision, the main a priori elements of mathematics—the group concept, the intuition of iteration, and the intuitive continuum—are each associated with what appear to be the fundamental mathematical structures. The group structure, the simply infinite structure, and continua are singled out as known a priori and as existing prior to the mathematical systems and constructions that instantiate them, which the a priori structures make possible.

A question about the consistency of my interpretation can now be addressed more clearly. The priority and independence of form over matter aligns Poincaré with the *ante rem* "structure first" view. However, in Shapiro's taxonomy, *ante rem* structuralism appears only as a form of realism,[34] and Poincaré was anti-realist about mathematics. (Despite his structural realism about natural science, Poincaré's appeal to mathematical intuition, his claims to defend Kant, and his views on mathematical existence all show this.)[35] Is this consistent? Is the structures-first, *ante rem* view consistent with the semi-Kantian constructivist view of mathematics that Poincaré defends? How can structures exist prior to, and/or independent of, mathematical objects and systems for an anti-realist?

One way to understand Poincaré is that the priority of structures to their objects and systems is merely epistemic and not ontological.[36] After all, as a constructivist, the ontology of mathematics will be constrained by its epistemology. If so, if the priority of structures is just epistemic, then one may object that his conception of structure better fits the eliminativist view than the *ante rem* view, since it is the main anti-realist alternative in the basic structuralist taxonomy. What about this alternative?

[34] While this is not asserted, the only type of *ante rem* structuralism he discusses is "Platonic."

[35] For example, he argues repeatedly against the existence of actual infinities because infinity just means there is "no reason for stopping" the generation of elements in a set.

[36] This appears to be Heinzmann's inclination (2014).

On the one hand, the emphasis on epistemology is right. The structures that are "prior" in Poincaré's philosophy are grounded in a priori intuitions and concepts, which, of course, lie more in the category of epistemology than ontology. On the other hand, eliminativism is the view that structures don't exist at all, that talk of "structure" is a mere manner of speech. And this does not fit Poincaré's views about mathematical structures. For him, structure is the core of the subject matter of mathematics; structure is the most important form of mathematical existence; indeed, as we saw above, focusing on the matter, or specific mathematical systems, rather than the form is "superficial" to him. This directly opposes the eliminativist view, according to which objects and systems may exist but structures do not.

Admittedly, Poincaré's rhetoric can be confusing. Intuitions and concepts do seem to concern epistemology; for example, the intuitions of iteration and continuity are "faculties" that enable the construction of simply infinite systems and physical and mathematical continua. Yet, for Poincaré intuition can also have a realist "feel."

> It is the intuition of pure number, that of pure logical forms, which illuminates and directs those we have called analysts. This it is which enables them not only to demonstrate, but also to invent. By it they *perceive at a glance* the general plan of a logical edifice. (1900, VI, 1020, my emphasis)

Here intuition is presented as like a telescope; it "illuminates"; it enables us to "perceive at a glance" things we couldn't otherwise perceive—pure logical forms, or structures. It is still epistemic, but it is articulated in terms of the ontology to which it provides access, rather than the activities by which we construct that ontology. And this may encourage a picture of the ontology as independent, or "out there." That is, a telescope does not create the objects we see with its aid;[37] thus, this way of describing intuition may encourage a similar view of mathematical structure—a more realist view of structure than would be consistent with constructivism. There are similar remarks about geometric intuition: because mathematics includes spaces of more than three dimensions, "there is surely *an intuition about the* continua of more than three dimensions" ([1913] 1963, 42–43, my emphasis). Finally, Poincaré explicitly says that the group structure "exists" prior to its instances. Recall that he says Helmholtz thinks the form of a group is posterior to, or dependent on, the matter, whereas in contrast, for him "the form *exists* before the matter" (my emphasis). At points like this, the structures of interest seem to preexist in a more ontological, rather than a merely epistemic, way.

[37] *Pace* a more strident form of scientific nominalism than would suit most, including Poincaré.

Is such talk just sloppy? Perhaps. To interpret Poincaré as consistent we must start with his mathematical anti-realism. That much is clear. Given this, *no* mathematical objects exist in a mind-independent sense, not even the fundamental structures. The "existence" of the group and intuitive structures is therefore something like existence in the minds of mathematicians. The priority in each case—a priori intuition and the a priori group concept—can be understood as a kind of mental template that enables specific instantiations, or constructions, in mathematical practice. So the priority of structures seems consistent with constructivism as long as "priority" and "existence" can be interpreted as reliant on minds (or finite thinking beings).

Furthermore, though they are generally associated with each other, *ante rem* structuralism does not require metaphysical realism. Note that Shapiro adds "Platonic" to the label "*ante rem* structuralism." This implies that the *ante rem* aspect—priority—does not entail Platonism, or realism; otherwise "Platonic" would be redundant. And this, in turn, implies that there can be a non-realist version of *ante rem* structuralism too.

That is, the term "*ante rem*" just indicates the priority in existence/reality of a general to its particulars. This is consistent with constructivism. Though for constructivists no mathematical entities exist absolutely independent of the minds and activities of mathematicians, some things can exist prior to others. Some templates can be required for some constructions, and some constructions can be required for others. An anti-realist version of the priority of structure would precisely fit Poincaré's conception of the a priori elements in mathematics—the elements that I have highlighted in my interpretation of his structuralism.

Of course, as mentioned, the taxonomy with which we began, and which is fairly common in the literature, is incomplete. Poincaré is not a Platonic *ante rem* structuralist because he is not a Platonist. He is not an *in re* structuralist because for him structures are not posterior to and dependent on systems; rather they are prior to and independent of the systems instantiating them. And he is not an eliminativist about structure because he believes that they—as well as many other mathematical objects—do exist (though we may have to do mathematics to make them exist). Eliminativism cannot be the only option for a structuralist who rejects realism about mathematical existence. Constructivist versions of both the *ante rem* and the *in re* views therefore seem coherent options.

4. Concluding Thoughts

Like many others at the time, Poincaré endorsed the basic methodological structuralist view. What's unusual for his time is that he expressed further philosophical views about the metaphysics and epistemology of mathematical structures. I have attempted to articulate these views and to situate them with respect to some of his other main commitments. In particular, I have argued that Poincaré's views about structure should be understood in a "strong" *ante rem* way, and that they are, nevertheless, consistent with his general constructivist approach to mathematics.

As a constructivist, Poincaré endorses a close connection between the epistemology and the ontology of mathematics. So his philosophy does not permit the same type of independence that one sees in traditional Platonism—that is, independence of mathematical reality from the minds and activities of mathematicians. But this is not the type of independence that characterizes *ante rem* structuralism. *Ante rem* structuralism only requires that structures be independent of, and prior to, their instances, which is exactly what Poincaré asserts.

For Poincaré, the form exists before the matter in that—to speak crudely—we need the form in order to cognize the matter. Though no mathematical entities or structures exist absolutely independent of the work and minds of mathematicians, the fundamental structures—the group structure, the natural number structure, and the continuum—have *more* independence than the domains they yield. In other words, they are prior to their instantiating systems.[38]

The fundamental mathematical structures, given by a priori concepts and intuitions, are thus, for Poincaré, a kind of cognitive blueprint necessary for conceiving and instantiating mathematical systems. Systems are the result of definitions; but both the definitions and our understanding of what they yield are guided by our "blueprints." The intuition of indefinite iteration guides our cognition of simply infinite systems as well as our inductive inferences about them. The intuitive continuum "enables" us to define physical and mathematical continua and to work with them in mathematics and science. The a priori group concept is a "form" that exists prior to the matter of any group, providing a unifying mathematical ideal and a foundation for geometry and group theory. In each of these cases the form "exists" prior to the domains, or systems.[39] Additionally, structures are more significant than matter or particular systems, which Poincaré dismisses as "superficial." This is the sense in which structures "preexist" for Poincaré: in the mind as an a priori intuition of structure, or as an a priori concept, rather than as independently existing Platonic reality.[40] Provided we agree that *ante rem* structuralism does not require Platonism, we can appreciate Poincaré's "constructivist *ante rem* structuralism"—a view that fits Shapiro's *ante rem* category, minus the "Platonism" typically attached to it.

[38] And possibly other, less fundamental, structures.

[39] Indeed, epistemologically speaking, each of these forms (the fundamental structures) are a priori, not just relatively prior.

[40] Whose mind? Interestingly not just humans for Poincaré; he implies that at least some a priori elements of mathematics (concepts, intuitions) are common to all finite beings who can conceive of infinity or space ([1902] 1952, p. 39; [1908] 1982, 427–428).

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