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Explication as Elimination: W. V. Quine and Mathematical Structuralism

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W. V. Quine has long been recognized as an important influence on the development of mathematical structuralism. Stewart Shapiro, for instance, uses the following remark from Quine's "Ontological Relativity" as the epigraph to his own structuralist treatise, *Philosophy of Mathematics: Structure and Ontology* (1997):[1]

> Expressions are known only by their laws, the laws of concatenation theory, so that any constructs obeying those laws . . . are *ipso facto* eligible as explications of expression. Numbers in turn are known only by their laws, the laws of arithmetic, so that any constructs obeying those laws—certain sets, for instance—are eligible in turn as explications of number. Sets in turn are known only by their laws, the laws of set theory. (Russell 1919, 44)

This statement is certainly clear in its structuralist commitments, but, taken out of context, its overall philosophical aims are far less clear. Many commentators have simply taken Quine to be part of that tradition of mathematical structuralism associated with Paul Benacerraf, stemming from his classic article "What Numbers Could Not Be" ([1965] 1983).[2] Here, Benacerraf argues that since structuralism about the natural numbers opens the way to a variety of mutually incompatible theories of what the numbers are, the conclusion to draw is the numbers are not objects at all. In one way or another, modern structuralists typically aim to respond to Benacerraf's challenge, to in some sense answer the question, what then are the numbers? We might also take Quine to be attempting to answer this question, but this seems potentially contrary to his naturalism if the question and answer are construed in a robustly metaphysical way. Furthermore, assimilating Quine to this tradition ignores that the beginnings of his structuralism

[1] In addition to Shapiro's work, see Parsons (1990, 2004); and Resnik (1997).

[2] Quine cites Benacerraf's paper in "Ontological Relativity," agreeing with the idea that arithmetic is all there is to the numbers, but also remarking that Benacerraf's "conclusions differ in some ways from those I shall come to" (1969b, 45 n. 9).

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can be found already in his earliest work, his 1932 dissertation, "The Logic of Sequences," some 30 years before Benacerraf's article. I will argue instead then that Quine's structuralism is much better situated and understood within the context of an early form of structuralism, specifically the structuralism Russell put forward as part of his program for scientific philosophy. While there is much diversity among the views of the early structuralists (as there is also among contemporary structuralists), which include also Dedekind and Carnap, one thing that unites them is the rejection of a more metaphysical view of mathematics and of structures more generally. They all put forward views of mathematics that, in a sense, answer only to mathematics itself. The basic idea here is that all an account of mathematical objects requires is that the entities—whatever they are—that serve as these objects satisfy the relevant postulates and theorems. Here we can see how Quine's early work in the foundations of mathematics leads in a natural way to the more general naturalism of his later philosophy.

In what follows, I will look at the development and motives for Quine's particular brand of mathematical structuralism. I will argue that Quine, unlike many contemporary mathematical structuralists, does not appeal to structuralism as a way of accounting for what the numbers *really* are. Instead, he denies the very conception of analysis that gives rise to such philosophical projects, that is, a conception of analysis that aims to divulge some deeper hidden extra-scientific metaphysical reality.[3] In this way, I see Quine's philosophy as firmly rooted in the tradition of scientific philosophy and its critical attitude toward more metaphysical varieties of philosophizing. The tendency to treat Quine's philosophy as part of the contemporary analytic scene, I think, misconstrues the radical nature of his views and its deep connections to the tradition of scientific philosophy, starting with Russell and running through to Carnap and, then, culminating in Quine. The structure of this chapter is as follows. In section 1, I provide a brief account of Russell's structuralism with a particular emphasis on its anti-metaphysical motivations. Here, I focus on Russell's work of the 1910s as this is where his own particular version of scientific philosophy emerges most clearly.[4] It is also the period in which he wrote his *Introduction to Mathematical Philosophy*, probably the text that Quine most often cites as inspiring his own commitment to structuralism. In section 2, I present the beginnings of Quine's structuralism, arguing that it emerged from his early and careful engagement with Russell's work in the foundations of mathematics. In section 3, I move to Quine's mature view. As we saw already, Quine's structuralism is often traced to his 1969 "Ontological Relativity." But I turn instead to his 1960 *Word and Object*,

[3] Where science, for Quine especially, includes mathematics.

[4] I think there are also structuralist aspects in Russell's earlier work as well, though they may differ in key ways from the view put forward in the 1910s.

as it is here that we find his most detailed discussion of his structuralism during this period. To use the taxonomy of Reck and Price, Quine's structuralism here is of the relativist variety: there are many models that satisfy the structural properties of mathematical objects, and the relativist structuralist simply chooses one of these models as, for example, the natural numbers. A different model could have been chosen, but so long as the choice is a consistent one, no conflict arises. For most relativist structuralists, Quine included, set theory provides the model (Reck and Price 2000, sec. 4).[5] Finally, in section 4, I argue that Quine's appeal to structuralism largely stands apart from the concerns of contemporary structuralism stemming from Benacerraf and his challenge that numbers are not objects.

1. Scientific Philosophy and the Russellian Background

Since I aim to situate Quine's structuralism in the context of Russell's program for scientific philosophy, let me begin by very briefly characterizing the tradition of scientific philosophy as it began to emerge in the second half of the 19th century.[6] The terminology "scientific philosophy" began to appear in the literature in the mid-1800s in reaction to the very speculative metaphysics of post-Kantian idealism and its attempts to distinguish the methods and aims of philosophy from those of the sciences. Alan Richardson emphasizes two aspects, in particular, to characterize this movement: first, a critical attitude toward metaphysics, sometimes extending to philosophy as a whole; and second, a cooperative spirit between philosophy and the sciences (1997, 426–427). This latter feature arose largely in reaction to philosophy's attempts during the 19th century to distinguish itself from the sciences by following artistic or religious models for philosophizing. This latter aspect is apparent from the start in Russell's work in the philosophy of mathematics. He frequently appeals to the results of mathematicians such as Peano and Cantor and urges that philosophers engaging in the philosophy of mathematics study the most up-to-date foundational work on the subject. Similarly, I think this characterization would be uncontroversial for Quine's work as well, taking naturalism as the central tenet of his philosophy. Indeed, I think most contemporary analytic philosophers would grant that philosophy should be done in cooperation with the latest results of science. The

[5] I think this is the best characterization of Quine's position, and I think it is also how Quine understands Russell's position in the 1910s. Reck and Price do point out a problem for relativist structuralism in that it seems that the objects of the basic theory, in most cases sets, are treated differently than the other objects of mathematics. Unlike, say, the numbers, the sets are not eliminated in favor of some other structure. I am not sure that Quine would feel the force of this objection. As we will see in section 16.4, Quine thinks that all objects—sets, numbers, atoms, tables, chairs, etc.—are given only by their structural properties.

[6] On the tradition of scientific philosophy, Friedman (2012) and Richardson (1997).

former characterization, however, is one that distinguishes the earlier tradition of scientific philosophy from much of contemporary analytic philosophy. And so it is this one that I will focus on throughout this chapter.

In the mid-1910s Russell explicitly put forward his program for scientific philosophy, urging that philosophy take up a scientific methodology so as to yield to philosophy that kind of progress already found in the sciences. He envisioned groups of independent researchers, each focusing on their own specialized research so that philosophy might proceed piecemeal. He diagnosed philosophy's floundering as rooted in its striving for a single grand system of the world. Russell proposed instead that

> The essence of philosophy . . . is analysis, not synthesis. To build up systems of the world, like Heine's German professor who knit together fragments of life and made an intelligible system out of them, is not, I believe, any more feasible than the discovery of the philosopher's stone. What is feasible is the understanding of general forms, and the division of traditional problems into a number of separate and less baffling questions. "Divide and conquer" is the maxim of success here as elsewhere. (2004b, 87)

Along with the rejection of such grand systematizing came also a skepticism toward more metaphysical approaches to philosophy.[7] For example, Russell considers the common-sense belief in the existence of permanent, rigid bodies such as tables, chairs, stones, and such as "a piece of audacious metaphysical theorizing; objects are not continually present to sensation, and it may be doubted whether they are there when they are not seen or felt" (1993, 107).[8] Elsewhere, he compares this assumption as akin to a Kantian *Ding an sich*.[9] He thinks such assumptions introduce unnecessary doubt into philosophy[10] and instead,

[7] This emerges as a definite theme in Russell's work of the 1910s. Russell's desire to emphasize this new focus perhaps also explains his retitling of his 1901 "Recent Work on the Principles of Mathematics" to "Mathematics and the Metaphysicians" (2004a) for its 1918 reprinting. There are many ways that metaphysics might be characterized. In this chapter I will focus on the idea that metaphysics divulges some sort of hidden reality that is in some way more real than the reality described by the natural sciences or, in this case, mathematics. I should add that Russell himself leaves open the possibility of metaphysics from within scientific philosophy (see, for example, 2004c, 127). I am emphasizing the strand of his thought that rejects what he refers to as "traditional metaphysics." Similarly, I am emphasizing the anti-metaphysical strand of Quine's thought. But Quine's view parallels Russell in also leaving open a scientifically acceptable metaphysics. Certainly, his aim of "limning the most general traits of reality" (1960, 161) has a metaphysical ring to it. Indeed, I think Quine's philosophy could be accurately described as the naturalizing of metaphysics. But we might then wonder whether this is metaphysics in any sense that a more traditional metaphysical philosopher would accept.

[8] *Our Knowledge of the External World* is scattered with remarks such as this, as are his other works from this period. For other examples see pp. 111–12, 134.

[9] See, for example, Russell 1993, 92. We will later see that Quine follows Russell in his motivations for structuralism here.

[10] See for example, Russell 1993, 134.

recommends trying to find constructions out of less dubious entities. He sums up this view as his "supreme maxim in scientific philosophizing": "*Wherever possible, logical constructions are to be substituted for inferred entities*" (2004c, 121).

Still, we might wonder how we can be assured that we have given an appropriate logical construction to serve the role of the desired object. To this point, Russell responds,

> Given a set of propositions nominally dealing with the supposed inferred entities, we observe the properties which are required of the supposed entities in order to make these propositions true. By dint of a little logical ingenuity, we then construct some logical function of less hypothetical entities which has the requisite properties. This constructed function we substitute for the supposed inferred entities, and thereby obtain a new and less doubtful interpretation of the body of propositions in question. (2004c, 122)

This is just the sort of structuralism about mathematics that he would go on to describe in his 1919 *Introduction to Mathematical Philosophy*, a text Quine read and often cites as inspiring his own structuralism.[11] In this later work, Russell presents Peano's axioms (this is "the body of propositions in question," in this case) for arithmetic and observes that any progression will satisfy them and also that any series satisfying the axioms is a progression. In this way, these axioms define the class of progressions.[12] Hence, any progression can be taken to do the work of the natural numbers in pure mathematics.[13] We simply identify the first object of the progression with zero, the second with one, the third with two, and so on. But since any progression will do, the members of the progression will not necessarily be the numbers as we ordinarily think of them. Russell says that they may be points in space, moments in time, or any other such infinite collection of objects: "Each different progression will give rise to a different interpretation of all the propositions of traditional pure mathematics; all of these possible interpretations will be equally true" (1919, 8–9). Russell later makes clear the philosophical import of this structuralism in explaining that similar constructions can also be carried out for geometry. He observes here that from a mathematical standpoint all questions about the "intrinsic nature" of geometric objects, such

[11] Quine cites this as one of the books that most influenced his philosophical direction (2008a, 328). Russell also makes this point with regard to mathematics in (1993, 209–210), another text that Quine read.

[12] Throughout this chapter, I use "class" and "set" interchangeably since this fits with Russell's and Quine's typical usage. I am not drawing the common distinction between sets and (proper) classes. Nor are Russell and Quine.

[13] Russell adds the condition that any such a progression should also be suited to applications of mathematics (1919, 9). Quine shows how this condition can easily be met by any progression (1960, 262–263).

as points, lines, and planes, can be put aside. Since points need be nothing more than what makes their axioms true, there is nothing further that needs to be said about them. All that a point requires is that "it has to be something that as nearly as possible satisfies our axioms, but it does not have to be 'very small' or 'without parts.' Whether or not it is those things is a matter of indifference, so long as it satisfies the axioms" (1919, 59).

Russell concludes his discussion by generalizing this account not only to the rest of mathematics but also to the rest of science, remarking: "This is only an illustration of the general principle that what matters in mathematics, and to a very great extent in physical science, is not the intrinsic nature of our terms, but the logical nature of their interrelations" (1919, 59).[14] By emphasizing the importance of structure here, Russell makes clear the form that his critical attitude toward more metaphysical approaches to philosophy takes. From his scientific standpoint, questions about the intrinsic nature of objects are dismissed. All that science demands of an object is that it satisfy the axioms or postulates of the relevant science. There is no deeper, mysterious essence about objects to be discovered; their structural properties are enough to meet the demands of scientific philosophizing.

2. The Beginnings of Quine's Structuralism

In his autobiography, Quine makes explicit Russell's influence, stating that he inspired Quine to philosophy and referring to *Principia Mathematica* as "the crowning glory" of his undergraduate honors reading (1985, 59). Quine's own 1932 dissertation, "The Logic of Sequences," was a reworking of roughly the first 400 pages of *Principia* and was written under the direction of Russell's coauthor, Alfred North Whitehead.[15] Here, too, Quine draws a further important connection to Russell, remarking on the philosophical significance of their respective works: "Outwardly my dissertation was mathematical, but it was philosophical in conception; for it aspired, like *Principia*, to comprehend the foundations of logic and mathematics and hence the abstract structure of all science" (1985, 85). Here we see already that Quine had absorbed Russell's point that what mattered most in mathematics and science was structure. In this section, I will sketch out

[14] Similarly, he makes this point concluding the previously quoted passage from "The Relation of Sense Data to Physics, remarking of his logical constructions: "This method, so fruitful in the philosophy of mathematics, will be found equally applicable in the philosophy of physics, where, I do not doubt, it would have been applied long ago but for the fact that all who have studied this subject hitherto have been completely ignorant of mathematical logic" (Russell 2004c, 122). As we will see in the final section, Quine, too, extends his structuralism to all of science.

[15] For background on the dissertation see Quine (1985, 84–86), as well as his preface to the version of the dissertation published in 1990 and also Dreben (1990).

the beginnings of Quine's structuralism and argue that it was motivated by the same critical attitude toward metaphysics that we saw in Russell. In this early period, however, Quine's views develop not with a focus on numbers but more generally on the nature of propositions.

Explicit philosophical discussions are largely absent from Quine's dissertation, but we do get some sense of the purpose that Quine's structuralism would later serve.[16] In particular, he places little weight on the intuitiveness of his system; there is no attempt to discover what the entities of mathematics *really* are. What matters is that the system be convenient for the mathematical work at hand. This emerges immediately in Quine's own account of propositions. To this end, he introduces the primitive operation of predication, which he describes as the binary operation upon function and sequence, yielding what he calls a *proposition*, expressed notationally by juxtaposing the two operands, φ and X, to get φX. This, Quine says, is all there is to a proposition: "Such is the manner in which propositions emerge in the present system. A proposition is for us a construct, a complex, wrought from a function and a sequence by the undefined operation of predication" (1990, 38).

Still, Quine recognizes that we might ask for more; we might reasonably think that a proposition is not just a formal construct:

> But, it may be asked, what sort of thing is this product of predication? From the official standpoint of our system, it is to be answered only that it is whatever predication yields; and predication is primitive. Unofficially, we may say that by a proposition we mean exactly what one ordinarily means by the term; and, from this standpoint, we may describe predication as that operation upon function and sequence which renders that latter argumental to the former and produces a proposition. In the terms of the present system, thus, the proposition is logically subsequent to the function and argument sequence which enter it. This treatment, however, is quite independent of metaphysical and epistemological considerations. It is altogether indifferent to the present system if function and argument be construed as abstractions which are, in some philosophical sense, subsequent to the proposition from which they are abstracted; just as it is irrelevant that, from a psychological standpoint, propositions are pretty certainly prior chronologically to function and sequences. *Nor, indeed, are we even concerned with maintaining that propositions are, in any absolute sense, logically subsequent to functions and sequences—mainly, perhaps, because we have little conception of what possible meaning such a statement might have.* The point is merely that is has proved convenient in the present system to frame

[16] Still, much Quinean philosophy can be pulled from the dissertation. For more on this topic, see Dreben (1990); and Morris (2015).

> our primitives in such a way that, for us, the proposition emerges as complex. (1990, 38–39; my emphasis).

Here we see again Quine's technical approach to more traditionally philosophical concerns, but he also highlights that there may be a number of philosophical concerns about propositions that he does not address. Quine, however, does not see this as a deficiency but rather as a benefit. Philosophical controversies over propositions, such as the ones he points out, are irrelevant to the technical development of propositions in his system. Engaging in these controversies, then, would only lead to the kind of stagnation that scientific philosophy had sought to avoid. Indeed, Quine's remark that his account is "independent of metaphysical and epistemological considerations" might be aimed at Russell himself. It is just these sorts of concerns—about the function-argument analysis of propositions, whether to take the components or the propositions as prior, and most generally, how to account for the very unity of a proposition at all—that leave much unresolved on the philosophical side of Russell's account.[17] Quine's conclusion, unlike Russell's, is simply that we have no firm ground to stand upon to even know exactly what question we are asking in these cases. As we see in the italicized sentence, Quine simply rejects that there is any absolute sense of what a proposition is; there is no question to ask about what propositions *really* are, aside from the account that Quine's logical system provides.

There is one further remark to note in this passage. He observes, "Unofficially, we may say that by a proposition we mean exactly what one ordinarily means by the term." Here, I think Quine gives the first hint that something like the structuralism we saw already in Russell will be conducive to Quine's own philosophical position. Indeed, in recognizing the difficulties Russell had with propositions, Quine sees structuralism as a solution, or better, a dissolution of the whole problem. Exactly what we mean ordinarily by a proposition is far from clear, but what is important in understanding Quine's view here is that he thinks there is some agreed-upon meaning or role that we ascribe to propositions.[18] And any technical entity that satisfies this role has equal claim to being a proposition.

Despite the apparent success of his account of propositions over previous accounts, we might still wonder why we should take them to be sequences. Here Quine brings us back to his emerging structuralism. In his 1934 *A System of Logistic*, the published version of his dissertation, he observes that Whitehead had emphasized the non-assertiveness of propositions, meaning that only in

[17] See in particular secs. 480–483 of *Principles of Mathematics* (1937), Russell's appendix on Frege's views. For useful commentary see also Hylton (1990, 336–338, 342–350), also his (2005); as well as Ricketts (2001), along with his (2002).

[18] This seems to be precisely what he later rejects about them, or more specifically, about the notion of analyticity, in his "Two Dogmas of Empiricism" (1980, 25), first published in 1951.

making a judgment does the proposition assert something true or false. For example, the proposition that this book is red does not assert that the book is in fact red. This only comes when a judgment is made. Quine then adds of his own account of propositions that "the doctrine of propositions as sequences stands in striking agreement with Whitehead's point of view; it presents a definite technical entity fulfilling just the demands which he makes of a proposition" (1934, 33). Significant here is not so much Quine's agreement with Whitehead but rather Quine's remark that he has provided a definite technical entity that fulfills the role that we expect propositions to play. The kind of ordinary meaning of propositions that he had in mind earlier is now made somewhat more precise. Propositions are those sorts of things that are potentially true or false, that serve as the postulates and theorems of a logical system, that can be manipulated in accordance with the rules of the system, etc. Again, we see Quine already in his earliest philosophy leaning toward the sort of structuralism found in Russell's *Introduction to Mathematical Philosophy*. Quine is not merely adopting the sort of formalistic or technical approach characteristic of much mathematical work. Rather, he takes such an approach to have philosophical consequences when embodied in a kind of structuralism. Here he eliminates traditional philosophical worries specifically over the true nature, or essence of, propositions. There is no deeper question to be asked about them than what role it is that they ordinarily play. If we can find a sufficiently clear technical entity that satisfies this role, there can be no further demand to make, aside from pragmatic concerns over whether that particular entity best suits the particular task at hand. While the paradigm case for such an account is no doubt the sort that Russell introduced with the numbers, Quine's account of propositions is very much in this same spirit. As we will see, this is the kind of clarificatory work that he would later identify as a paradigm of philosophical analysis (1960, sec. 53).

After the mid-1930s all positive talk of propositions drops out of Quine's work.[19] Perhaps foreshadowing his later attack on the analytic/synthetic distinction, he came to realize that talk of propositions lacked contexts that were "clear and precise enough to be useful" (1980, 25). Furthermore, he may have come to see that his technical replacements could be rendered less controversial by simply calling them what they were—sequences, sentences, or what have you. Still, this early work on propositions is important in setting up Quine for the sort of structuralism he would adopt in his mature philosophy. After all, there are many entities—numbers among them—crucial to science and in need of very much the kind of analysis Quine had offered for propositions in this early period.

[19] Quine's other significant work on propositions, also from 1934, is his "Ontological Remarks on the Propositional Calculus" (1976b). The discussion here is complimentary to both the dissertation and *A System of Logistic*.

3. Quine's Mature View

In light of more recent structuralist approaches to mathematics, which tend to respond directly in one way or another to Benacerraf's challenge, I hope in this and the next section to give us a better sense of what Quine's structuralism is both meant and not meant to do. From the perspective of contemporary structuralism, Quine's discussion may perhaps appear simplistic or inadequate. He never addresses many of the worries that we see in current discussions. I take it that this is intentional on his part as his structuralism is largely meant to deny certain kinds of philosophical worries. Quine is not, for example, trying to answer the question of what the numbers *really* are or more generally, what structures *really* are.[20] Rather, as we will see, he aims to dissolve rather than solve philosophical puzzles such as this one.

While Quine's claim in "Ontological Relativity" that the numbers are known only by their laws is perhaps his most explicit statement of a kind of mathematical structuralism, his most sustained discussion of his view occurs in *Word and Object*. We saw in section 1 that Russell's structuralism arose out of his urging of the analytic method as the right way to pursue scientific philosophy. Quine continues with this approach, adopting in section 53, "The Ordered Pair as a Philosophical Paradigm," a kind of structuralism as a general method for philosophical analysis. Here he describes a common situation where we have a term that is in some sense defective but that is also very useful to our theorizing. We must then somehow make sense of it, preserving its utility while removing its defectiveness. Quine looks to the ordered pair as a particularly clear case of just this phenomenon. Typically, we find this device in mathematics where it allows us to assimilate relations to classes by treating the relations as classes of ordered pairs (1960, 257). Its defectiveness readily appears when we try to give an account of what an ordered pair is. Referring to Peirce's nearly impenetrable account in terms of a mental diagram, Quine recommends instead, "We do better to face the fact that 'ordered pair' is (pending added conventions) a defective noun, not at home in all the questions and answers in which we are accustomed to imbed terms at their full-fledged best" (WO, 257–258). He then explains that mathematicians take the single postulate

(1) If $\langle x, y \rangle = \langle z, w \rangle$ then $x = z$ and $y = w$,

[20] I take it that many contemporary structuralists would agree on the first question, but it does seem that the discussion then just shifts the worries that arose around numbers to worries about the structures themselves.

to govern all uses required of the ordered pair. So we want a single object that will do the work of two and that satisfies this condition. The solutions, Quine observes, are many, with Kuratowski's rendering of $\langle x, y \rangle$ as $\{\{x\},\{x, y\}\}$ being among the most common. But Norbert Wiener's $\{\{x\},\{y,\varnothing\}\}$ serves the purpose equally well. It is straightforward to show that either of these classes satisfies postulate (1) (1960, 258–259).[21] This, Quine declares, is precisely what a philosophical analysis should do:

> This construction is paradigmatic of what we are most typically up to when in a philosophical spirit we offer an "analysis" or "explication" of some hitherto inadequately formulated "idea" or expression. We do not claim synonymy. We do not claim to make clear and explicit what the user of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words 'analysis' and 'explication' would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worth troubling about, and then devise a substitute, clear and couched in terms to our liking, that fills those functions. Beyond those conditions of partial agreement, dictated by our interests and purposes, any traits of the explicans come under the head of "don't-cares." (1960, 258–259)[22]

The analysis of the ordered pair is unusual only in that the condition of partial agreement can be made so explicitly and simply. Other cases of analysis will not be so straightforward, but on Quine's account, this is still ultimately what any such analysis is meant to accomplish.[23]

There is then no answer to which of these analyses of the ordered pair is the correct one. Any object satisfying (1) has equal right to being the ordered pair, and this, Quine says, is the general situation with any analysis, or explication. For

> *explication is elimination*. We have, to begin with, an expression or form of expression that is somehow troublesome. It behaves partly like a term but not enough so, or it is vague in ways that bother us, or it puts kinks in a theory or encourages one or another confusion. But it also serves certain purposes

[21] See, for example, Enderton (1977, 35–36). There are plenty of other equally good analyses of the ordered pair; Quine (1960) gives further examples on p. 260.

[22] "Explication" is of course Carnap's terminology; see, for example, *Meaning and Necessity* (1956, 7–8). Part of what I hoped to have shown in section 16.2 was that Quine had this notion already in place prior to any serious engagement with Carnap's work. A more general conclusion, which I have not argued for in this chapter, is that Quine's and Carnap's shared philosophical aims can be traced back to the common influence of Russell.

[23] Here is at least part of his rejection of the analytic-synthetic distinction. Quine just does not think that we have any idea of what the conditions of partial agreement should be for the analysis of analyticity.

> that are not to be abandoned. Then we find a way of accomplishing those same purposes through other channels, using other and less troublesome forms of expression. The old perplexities are resolved. (1960, 260)

In the end, the question of what an ordered pair is is dissolved when this troublesome notion is replaced by some clearer notion. And now to bring us more directly back to structuralism about the numbers, Quine goes on to say exactly this of Frege's analysis of numbers as well, citing Russell's *Principles of Mathematics* as his source. Here Quine presents the more typically philosophical question "What is a number?" and—just as Wiener and Kuratowski did for ordered pairs—we have Frege replacing these somewhat mysterious entities with the better-understood classes. On this account, for each number *n*, we identify it with the class of all *n*-membered classes (the seeming circularity here can be paraphrased away). Quine then observes that to object that classes have different properties from numbers is to make no objection at all. It is just to misunderstand the point of explication:[24]

> Nothing needs be said in rebuttal of those critics, from Peano onward, who have rejected Frege's version because there are things about classes of classes that we have not been prone to say about numbers. Nothing, indeed, is more logical than to say that if numbers and classes of classes have different properties then numbers are not classes of classes; but what is overlooked is the point of explication. (1960, 262, footnote omitted).

Furthermore, again like the ordered pair, this is just one of many ways of explicating numbers. Von Neumann and Zermelo offered other analyses. None are equivalent but all serve perfectly well as the numbers. Quine concludes that, as with the ordered pair, we can provide a condition that any explication of number must satisfy. Such a condition is provided by the notion of a progression, and any objects satisfying it will serve perfectly well as the numbers.

4. Quine and Modern Structuralism

I have been describing the development of Quine's structuralism, but let me now come back to the more general point I wanted to make about Quine's place in the history of analytic philosophy. I began with Russell so as to emphasize his influence on Quine's structuralism, and in particular, the critical attitude toward

[24] Russell does this as well (1919, 18–19).

metaphysics, characteristic of the scientific tradition of philosophy. We saw this with regard to propositions, where Quine showed how propositions could be rendered in terms of the sequences of his logical system. There was no worry here about whether these are really what propositions are. Sequences of a certain sort turned out to fulfill just the role required of propositions in his system. Quine's point was that there was no further demand to be made of them. Here, I stressed that this was a decidedly philosophical view on Quine's part. We see it now fully developed in his later work. What is to be emphasized here is again his rejection of certain philosophical questions—by eliminating problematic entities in favor of some that are better understood, Quine not so much solves as dissolves philosophical questions (1960, 260). The importance of elimination here cannot be stressed enough for properly understanding the significance and purpose of the remark with which we began this chapter, that "numbers . . . are known only by their laws, the laws of arithmetic" (1969b, 44). It is precisely on this point that I think Quine's position can be distinguished from much of what goes on in contemporary mathematical structuralism. Let me try to explain why.

Most of the contemporary discussion of mathematical structuralism has been set by Benacerraf's "What Numbers Could Not Be" ([1965] 1983). In it, Benacerraf famously concludes that the numbers cannot be objects (290). Since numbers, unlike other sorts of objects, have their requisite properties only in relation to the other numbers, we cannot give an account of any particular number short of characterizing the entire abstract structure of arithmetic. As he explains:

> The pointlessness of trying to determine which objects the numbers are thus derives directly from the pointlessness of asking the question of any individual number. For arithmetical purposes the properties of numbers which do not stem from the relations they bear to one another in virtue of being arranged in a progression are of no consequence whatsoever. But it would be only these properties that would single out a number as this object or that.
>
> Therefore, numbers are not objects at all, because in giving the properties . . . of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. . . .
>
> Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions. It is not a science concerned with particular objects—the numbers. The search for which independently identifiable particular objects the numbers really are (sets? Julius Caesars?) is a misguided one. ([1965] 1983, 291)

Benacerraf's remarks here illustrate how far Quine's view is from the concerns of much of contemporary structuralism. The discussion here tends to attempt a direct response to Benacerraf's conclusion. Participants in the dialogue either accept it and try to work out more precisely what it means for the numbers to not be objects; or they reject it and try to show how despite being recognized only by their structural properties, numbers still have a claim to being objects of a rather special sort.[25] For Quine, this entire discussion assumes too much from the start, resting on the uncritical assumption that we have some conception of an object ready to hand within which we can make sense of these two options. I will not be able to treat fully Quine's views on ontology and objecthood here, but let me try to give some better indication of how I think Quine sees the matter.[26]

Whereas Benacerraf assumes at the outset that the notion of an object is well understood and that the numbers are not instances of it, Quine does not. For Quine, we cannot assume as given that we know what will be among the objects and what will not. He takes ontology itself as a theoretical undertaking, one to be worked out in accord with the best science of our day. So as to where to draw the boundary between object and non-object, Quine responds,

> It is a wrong question; there is no limit to draw. Bodies are assumed, yes; they are the things, first and foremost. Beyond them there is a succession of dwindling analogies. Various expressions come to be used in ways more or less parallel to the use of the terms for bodies, and it is felt that corresponding objects are more or less posited, *pari passu*; but there is no purpose in trying to mark an ontological limit to the dwindling parallelism. (1981b, 9)

So our paradigm for an object might be bodies, that is, ordinary physical objects, but beyond this, there are just "dwindling analogies." We cannot simply rely on the notion of an object as given to us as fully understood. But then what are we to do about ontological questions? Should they just be rejected wholesale in the spirit of Carnap? No, as Quine continues:

[25] For the former view, I have in mind an eliminative structuralist such as Geoffrey Hellman. For his view see, for example, his (1989). I will not discuss his views further as, with their reliance on modal notions, I think they are far from anything that Quine would be willing to accept. For the latter view, I have in mind philosophers such as Michael Resnik or Stewart Shapiro. Reck and Price identify Resnik and Shapiro as both being "pattern structuralists"; that is, they are both committed to some version of the view that mathematics investigates patterns, and these are in themselves real objects. Shapiro's pattern structuralism is the more robust of the two, identifying the numbers with a sort of universal pattern (he calls his own view *ante rem* structuralism). Resnik also claims that numbers are patterns, but takes Quine's doctrine of ontological relativity more seriously and so does not identify the numbers with any one pattern. I am brushing over many subtleties in their views, but see Reck and Price (2000, sec. 7) for a more detailed summary.

[26] For a more detailed account of Quine's views, see Hylton (2004) and on abstract objects specifically see his (2007, 258–259).

> My point is not that ordinary language is slipshod, slipshod though it be. We must recognize this grading off for what it is, and recognize that a fenced ontology is just not implicit in ordinary language. The idea of a boundary between being and nonbeing is a philosophical idea, an idea of technical science in a broad sense. Scientists and philosophers seek a comprehensive system of the world, and one that is oriented to reference even more squarely and utterly than ordinary language. Ontological concern is not a correction of a lay thought and practice; it is foreign to the lay culture, though an outgrowth of it. (1981b, 9)

Contrary to Benacerraf, then, Quine thinks that the notion of an object itself and what it is to be ontologically committed to it stands in need of philosophical explication. Without providing some explicit criteria here, we cannot say whether or not numbers are to be counted among the objects. First of all, Quine tells us that for something to count as an object, we must have identity criteria for it, as summed up in his oft-quoted slogan, "No entity without identity" (1969c, 23; 1981a, 102). This tells what might be acceptable as an object, but it does not yet tell us if we are in fact committed to the existence of some particular object.[27] For example, surely we know the identity criteria for numbers, but the question here is whether we are in fact committed to the existence of numbers as objects. Clearly, we do talk of numbers as if they are objects, making claims such as "There is a number that is the successor of zero." But as we saw Quine point out, ordinary language is not a sure guide to ontological commitment.

Accepting that we cannot just read off of our everyday language what objects there are, Quine proposes a technical substitute. Using first-order quantification theory, Quine recommends that we regiment our scientific theory and then simply read off its ontological commitments by way of the universal and existential quantifiers, understood respectively as "for all objects *x*" and "there exists an object *x*." His solution to this quandary about objects is nicely summed up in another of his familiar slogans: "To be is to be the value of a variable" (1939, 708). Given this account, we now have a clear sense of what it means for an object to exist or not. So, for Quine, unlike Benacerraf, the numbers have every right to be considered objects alongside our ordinary physical objects so long as we are willing to countenance both as values of variables. Of course, we might reject Quine's criterion for ontological commitment, a possibility that he is well aware of. He welcomes other proposals, but to the extent that they do not capture the locution "there exists an object *x*," he sees them as giving no intelligible account of ontological commitment.[28]

[27] This criterion is closely tied into how Quine sees reification setting in. For a much more complete account of Quine's views here, again see Hylton (2004).

[28] See, for example, Quine's "Existence and Quantification," where he compares his objectual quantification with substitutional quantification (1969a, 103–108).

Benacerraf was driven to reject numbers as objects because of what he saw as some of their rather odd characteristics, chief among them that many different structures would do the work of the numbers. Again, this is clearly something that Quine is well aware of, noting many times, following Russell, that any progression will do. And here we see also the importance of elimination in Quine's account. To say that the numbers are some progression, for example, von Neumann's set-theoretic account, raises the question of why the numbers are this progression and not, for example, the one given by Frege or by Zermelo. On Quine's account of explication, we do not make such an identity. We have eliminated some apparent objects, not well understood, and replaced them with objects that are in some sense better understood. Out of habit or convenience, we refer to these sets as the numbers, but they are in the end just sets. These sets preserve whatever we found useful about numbers while pushing off any other features of the old numbers as "don't cares." There were of course other options for our explication, but as Quine observes, "Any progression will serve as a version of number so long as and only so long as we stick to one and the same progression. Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic" (1969b, 45).[29] The choice may be guided by certain pragmatic concerns. So in some other context we are equally free to choose a different analysis or explication, better suited to whatever that particular context requires (1960, 263). Here again we see the importance of not losing sight of Quine's point about explication being elimination. Whatever explication, or analysis, of numbers we opt for is all that is left of the numbers. There is no further independent question about whether we have correctly identified *the* numbers. The numbers have been eliminated in favor of some progression that has whatever features made the numbers worth explicating in the first place.[30] A failure to appreciate this aspect of Quine's account leads to the sort of worry Benacerraf identifies—which of the various progressions are the numbers really? For Quine, we might say, this is just a metaphysical pseudo-question (2008b, 401, 405).

[29] Note that the wording here is very much like the wording in the passage from *The Logic of Sequences* saying that there is no absolute sense of propositions short of some particular system (1990, 39).

[30] Again, Reck and Price place Quine's structuralism under the heading of relativist structuralism. The idea here is that there may be many structures that will serve as the natural numbers, and what we do is just pick one of them and stick with it. Reck and Price raise as a central question for relativist structuralism what we are to do about the basic level, the sets. Should not these also be treated in some structuralist way? It seems to me that Quine does have in mind also treating sets along structuralist lines. For example, in the quotation with which we began this chapter he remarks that it is not just numbers that are known by their laws but also sets. Indeed, as we will see, Quine thinks that in a sense all there is to any sort of object is its place in a theory. In this sense, I think Resnik correctly identifies Quine's position as "structuralism all the way down" (1997, 266).

So, on Quine's account the numbers are objects, but there is another line of thought that also treats the numbers as objects but that still seems at odds with Quine's account. Many prominent contemporary mathematical structuralists, chief among them Michael Resnik and Stewart Shapiro, agree with Quine that numbers are objects, but they also think there is something to what motivates Benacerraf to his conclusion: the numbers do somehow seem different from other sorts of objects; they do not seem to be objects in any ordinary sense. With this thought in mind, each in his own way tries to work out how the numbers might still be objects of a sort. Putting aside much detail, both embrace what Charles Parsons has identified as the incompleteness of mathematical objects.[31] In short, mathematical objects are incomplete in the sense that there are certain questions that we cannot answer about them since, as Benacerraf observed, they are given only by their relations within the entire structure of mathematics. So we seem to be at a loss about what the intrinsic nature of each number is; again, whether the numbers are in fact these sets or those.[32] Whereas Benacerraf indicated this as a problem for treating numbers as objects, Resnik and Shapiro just take this as characteristic of the particular kind of objects that the numbers are.[33]

We have already had a hint of Quine's response to this sort of worry about mathematical objects. His appeal to the quantifiers not only tells us what objects there are but is also univocal—Quine has no modes of being; there is only a single all-purpose notion of existence, applying to all objects indiscriminately.[34] This could be taken as a weakness of Quine's account; perhaps we would be better off recognizing somehow that the numbers, while still objects (contra Benacerraf), are unique in being identifiable only by their role in the structure of arithmetic as a whole. Quine surely recognizes differences among abstract objects, such as numbers, and the more ordinary concrete objects. In particular, he notes that we can learn terms for visible concrete objects by ostension, whereas this is not possible for terms for abstract objects (though more accurately he says that this difference is better reflected in the distinction between observation and theoretical terms). This, however, is an epistemological difference, rather than one reflecting a difference in kind among the objects themselves (1998, 402; 1981b, 16).

[31] Resnik explicitly adopts Parsons's terminology. Shapiro does not but attributes the appropriate characteristics to the numbers for them to be incomplete in Parsons's sense. See MacBride (2005) for a much fuller elaboration of this issue. While generally against, as we will see, characterizing mathematical objects, and abstract objects generally, as existing in some way differently from how concrete objects exist, Quine does not object to Parsons's technical work on incomplete existence. He just thinks the resulting theory not open to ontological assessment (1998, 400).

[32] See MacBride for this characterization (2005, 564).

[33] It should be noted that both Resnik and Shapiro describe themselves as Quineans of a sort. It may be that in light of their attempts to respond more directly to Benacerraf's challenge, Quine might have re-evaluated his own view on the matter. I will not undertake this task here on Quine's behalf, though I think it a worthwhile undertaking on the whole.

[34] Hylton emphasizes this point; see his (2007, 258; see also 303).

But still, what about the seemingly unique structural aspect of numbers? Here, too, Quine would be unconvinced, for this does not seem to be a unique feature of numbers after all, as shown by his doctrine of ontological relativity. He illustrates this most straightforwardly with what he calls proxy functions, where such a function maps our old objects onto some new objects (1969b, 55–61; 1981b, 19). For example, we might have the function *f* taking each object to its spatiotemporal complement *f*(*x*). With the predicates and terms appropriately adjusted, evidential support for the old and new theories remains the same, and so they are empirically indistinguishable. Here we have a version of what Quine calls his "global ontological structuralism" (2008b, 405):

> Structure is what matters to a theory, and not the choice of its objects. F.P. Ramsey urged this point fifty years ago, arguing along other lines, and in a vague way it had been a persistent theme also in Russell's *Analysis of Mind*. But Ramsey and Russell were talking only of what they called theoretical objects, as opposed to observable objects.
>
> I extend the doctrine to objects generally, for I see all objects as theoretical. (1981b, 20)

As he sums up his point, "Save the structure and you save all" (2008b, 405).

The point I wish to draw from this last discussion is that Quine will not be tempted to describe mathematical objects as incomplete. His global structuralism shows that there is nothing unique about the structural aspects of mathematical objects; much the same can be said of concrete objects. For Quine, in a sense, either all objects are incomplete or none are. No special trait of mathematical objects is picked out by their apparent incompleteness. As Quine describes his own view: "My own line is a yet more sweeping structuralism, applying to concrete and abstract objects indiscriminately" (2008b, 402). Resnik is, I think, then correct in describing Quine's view as "structuralism all the way down" (1997, 266). Resnik, however, wishes to contain his own structuralism so that it applies only to mathematical objects:

> By contrast, mathematical structuralism, including my own, finds its roots in the philosophical remarks of Dedekind, Hilbert, Poincaré, and Russell, and Paul Benacerraf's provocative thoughts on the multiple reduction of arithmetic to set theory. It takes the thesis that mathematical objects are incomplete ("known only by their laws") as a datum and tries to explain it, and consequently it does not go as far as Quine's. (1997, 267)

Shapiro, in his own way, joins Resnik in this view. Now, I am not claiming that Quine would reject any of the technical work that Resnik and Shapiro have contributed toward a mathematical theory of structures. What worries Quine are the motivations—that there is a desire on the part of structuralists such as Resnik and Shapiro to preserve some special status for mathematics (not unlike Carnap's attempt to declare mathematics analytic).[35] We see this here in Resnik's remark that he takes the incompleteness of mathematical objects as a datum to be explained by structuralism. This is precisely the kind of assumption that Quine's doctrine of ontological relativity, and his associated structuralism, denies. He describes his own global structuralism as coming from his naturalism—that is, from science itself—and its rejection of "the transcendental question of the reality of the external world—the question whether or in how far our science measures up to the *Ding an sich*" (1981b, 22).[36] He does not begin by assuming that mathematical objects are unique in some way. Ontological relativity shows mathematical objects no more, and no less, incomplete than ordinary concrete objects are.

We might, however, think such an unorthodox view to be in tension with Quine's professed realism.[37] He thinks not:

> Naturalism itself is what saves the situation. Naturalism looks only to natural science, however fallible, for an account of what there is and what what there is does. Science ventures its tentative answers in man-made language, but we can ask no better. The very notion of object . . . is indeed as parochially human as the parts of speech; to ask what reality is *really* like, however, apart from human categories, is self-stultifying. It is like asking how long the Nile really is, apart from parochial matters of miles or meters. Positivists were right in branding such metaphysics as meaningless. (2008b, 405)

Naturalism allows no deeper insight into reality than what will tolerate Quine's doctrine of ontological relativity. His global structuralism, then, just tells us what can be coherently said of objects unless we allow for some form of mystical

[35] The situation is much like that with regard to the analytic/synthetic distinction. Quine saw no flaws in Carnap's technical work. It was the underlying philosophical motivations that worried him: "In recent classical philosophy the usual gesture toward explaining 'analytic' amounts to something like this: a statement is analytic if it is true by virtue solely of the meanings of words and independently of matters of fact. It can be objected, in a somewhat formalistic and unsympathetic spirit, that the boundary which this definition draws is vague or that the definiens is as much in need of clarification as the definiendum. This is an easy level of polemic in philosophy, and no serious philosophical effort is proof against it. But misgivings over the notion of analyticity are warranted also at a deeper level, where a sincere attempt has been made to guess the unspoken *Weltanschauung* from which the motivation and plausibility of a division of statements into analytic and synthetic arise" (1976a, 138).

[36] Recall this is one of the ways that Russell described the aim of his structuralism.

[37] For more on the radical nature of Quine's views here, see again Hylton (2004, especially sections IV and V).

insight into the *true* nature of reality. Here, I have been describing Quine in terms that may seem more appropriate to a discussion of Carnap, and in this passage, we see Quine himself doing so. While I do want to stress, much more than is usually done, the significant continuities between Quine and Carnap, especially as part of a tradition of scientific philosophy stemming from Russell, I do not want to abolish the differences. And nor does Quine, as he then explains. Where the positivists went wrong was in trying to deny ontological questions altogether (2008b, 405). Still, Quine's own countenancing of such ontological questions, and in particular, his structuralism, is not a return to a more traditional brand of metaphysical theorizing, as he concludes:

> My global structuralism should not . . . be seen as a structuralist ontology. To see it thus would be to rise above naturalism and revert to the sin of transcendental metaphysics. My tentative ontology continues to consist of quarks and their compounds, also classes of such things, classes of such classes, and so on, pending evidence to the contrary. My global structuralism is a naturalistic thesis about the mundane human activity, within our world of quarks, of devising theories of quarks and the like in the light of physical impacts on our physical surfaces. (2008b, 406)

And here Quine brings us back to Russell. What matters most in the ontology of mathematics, and in the sciences more generally, is not the intrinsic nature of the objects but rather their structural relations to one another.

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