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Introduction and Overview

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1. Structuralism in the Philosophy of Mathematics: A Brief History

The core idea of mathematical structuralism is that mathematical theories, always or at least in many central cases, are meant to characterize abstract structures (as opposed to more concrete, individual objects). Thus, arithmetic characterizes the natural number structure, analysis the real number structure, and traditional geometry the structure of Euclidean space. As such, structuralism is a general position about the subject matter of mathematics, namely abstract structures; but it also includes, or is intimately connected with, views about its methodology, since studying such structures involves distinctive tools and procedures. The goal of the present collection of essays is to discuss mathematical structuralism with respect to both aspects. And this is done by examining contributions by a number of mathematicians and philosophers of mathematics from the second half of the 19th and the early 20th centuries.

In English-speaking philosophy, structuralist ideas have played a role for a while; but the current discussion of structuralism, as a main philosophical position, started in the 1960s. A crucial article often referred to in this context is Paul Benacerraf's "What Numbers Could Not Be" (1965). This article was a reaction against the view, dominant at the time, that numbers and other mathematical objects are all sets. For example, the natural numbers are the finite von Neumann ordinals familiar from Zermelo-Fraenkel set theory; and the real numbers are Dedekind cuts constructed in a set-theoretic way. According to Benacerraf, this kind of position misrepresents mathematics by leaving out its structuralist aspects. Beyond Benacerraf, there were other reactions against such a set-theoretic, foundationalist orthodoxy. For example, in Hilary Putnam's article "Mathematics without Foundations" (1967), a form of if-then-ism for mathematics was suggested instead (more on both subsequently).

It took until the 1980s for the debates about mathematical structuralism to really pick up steam. The main impetus came from a number or writings by Michael Resnik, Stewart Shapiro, Geoffrey Hellman, and Charles Parsons (cf. Resnik 1981, 1997, Shapiro 1983, 1997, Hellman 1989, 1996, Parsons 1990, 2009, among

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others). While Benacerraf had suggested thinking of the natural numbers, say, as an "abstract structure," distinct from all set-theoretic systems, he remained noncommittal and somewhat vague about the nature of such structures. Resnik and Shapiro took that notion more seriously, suggesting that we should think of them as abstract "patterns." In Shapiro's hands, especially, the patterns were then conceived of as a novel kind of abstract entity, to be described and studied in a general "structure theory." While more focused on epistemological issues, Resnik resisted reifying the relevant patterns. But for both Resnik and Shapiro, particular mathematical objects, such as specific natural or real numbers, are "positions" in such structures. In addition, Charles Parsons developed a distinctive variant of such a "structuralist view of mathematical objects" (more on the differences soon).

Shapiro characterized his position further in a twofold way: as a form of "realism"; and as "*ante rem* structuralism." What exactly realism amounts to in this context is a difficult, slippery question. But at a minimum, it involves taking mathematical statements, such as 2 + 3 = 5, at face value, in the sense that "2," "3," and "5" are seen as singular terms referring to abstract objects to which we ascribe properties, etc. Shapiro called his position *ante rem* since he took his abstract structures to be "ontologically independent" of their more concrete "instantiations," including set-theoretic ones. Often it is assumed in this context that the *ante rem* aspect directly implies the "realist" one. But as we will see later, this is misleading and wrong in general; the two can be, and have at times been, separated. Moreover, Parsons explicitly distinguishes further metaphysical claims involving "realism" from the basic structuralist conception he accepted. This means that one can be a "pattern structuralist" without being a realist, except in the minimal sense already mentioned. (Both points will matter later.)

In widely adopted terminology, Parsons also distinguished between "eliminative" and "non-eliminative" forms of structuralism (Parsons 1990). According to "non-eliminative structuralism," abstract structures are accepted, or postulated, as *sui generis* objects (different from other kinds of objects, including set-theoretic systems). Shapiro's *ante rem* structuralism is a main example; but Resnik's and Parsons's forms of structuralism are others. A paradigmatic example of "eliminative structuralism" is Geoffrey Hellman's "modal structuralism," intentionally devised to be a "structuralism without structures" (Hellman 1996). Building on Putnam's if-then-ism (and earlier ideas in Russell), Hellman proposed to interpret every mathematical statement as having a modalized if-then form. For example, "2 + 3 = 5" has the form "Necessarily, for all models M of the Dedekind-Peano axioms, $2_M + 3_M = 5_M$," where 2_M, 3_M, and 5_M are what "play the roles" of 2, 3, and 5 in the model M, etc. (cf. Reck and Price 2000 for details). Along such lines, structures seen as abstract objects are "eliminated"; we don't need to assume their existence. In fact, Hellman's position is "eliminativist" in a very strong sense, since reference to abstract objects is avoided altogether. Instead, mathematics becomes the study of certain possibilities (the Dedekind-Peano axioms, say, have to be possible) and necessities (general if-then statements such as the preceding example).

Shapiro and Hellman have worked out their positions in great detail. For both, this includes distinguishing "algebraic" mathematical theories, such as group theory, lattice theory, and topology, from "non-algebraic" ones, such as the theories of the natural numbers, real numbers, and sets. With respect to the latter, we are dealing with categorical (or at least quasi-categorical) theories, which means that all their models are isomorphic (up to the height of the set-theoretic hierarchy in the case of Zermelo-Fraenkel set theory). It is such theories to which their accounts are meant to apply primarily. For the non-categorical ones a more indirect approach is used. Beyond Shapiro's and Hellman's positions, other versions of structuralism have been proposed, and they usually involve a similar distinction. We already mentioned Resnik's and Parsons's positions on the non-eliminative side; Charles Chihara's is another example on the eliminative side (Chihara 2004); and we will encounter more later.

Since the 1980s, the debates about structuralism in mathematics have been extended in other respects too. Three trends stand out especially. First, some comparative studies have been offered (Hellman 2001, 2005; Cole 2010; Shapiro 2012; also Reck and Price 2000, on which we will build). One of their results is that Shapiro's, Hellman's, and similar positions rely, at bottom, on the assumption of a kind of "coherence" for the mathematical theories at issue, besides their categoricity. (In light of Gödel's theorems, this replaces provable consistency as a basic requirement for mathematics.) Somewhat surprisingly, such positions are thus very similar in a basic respect (which, among others, puts the "realism/anti-realism" distinction into a new light). A second development, from around 2000 on, has been to further probe certain features of Shapiro's structuralism especially, but also of other forms of non-eliminative structuralism. One example is that "positions" in structures are taken to be "ontologically dependent" on the whole structure. But how exactly is that to be understood? (Cf. Linnebo 2008, among others.) Another example is that, according to Shapiro's and similar positions, "structurally indistinguishable" objects should be identified. Yet that leads to problems in the case of "nonrigid" structures (with nontrivial automorphisms), such as the system of complex numbers (see, e.g., Keränen 2001; Leitgeb and Ladyman 2008; and Shapiro 2008).

A third main development since the 1980s has been the introduction and promotion of category-theoretic forms of structuralism, by Steve Awodey, Elaine Landry, Jean-Pierre Marquis, Colin McLarty, and others (cf. Awodey 1996, Landry 2009, McLarty 2004, Marquis 2008). While all the versions of structuralism we have discussed work, in one form or another, with first-order logic and set theory (perhaps modified slightly, e.g., in terms of Hellman's modal logic), category theory involves a radical shift away from that framework. This affects the way in which "structuralist" ideas are implemented. Roughly speaking, in categorical language only "structural properties" are expressible (see Landry and Marquis 2005, Korbmacher and Schiemer 2018 for more); and crucial features involving them can be highlighted further, e.g., in terms of "universal mapping properties." This makes the approach "structuralist" in a distinctive, very basic way. But category theory is also taken to be an alternative, significantly different "foundational" framework for mathematics (which has led to debates about the notion, or notions, of foundations involved). For both reasons, "categorical" versions of structuralism are hard to compare with those mentioned earlier. That being said, category theory is in line with an important shift in mathematical methodology that emerged in the 19th and early 20th centuries, and investigating that shift further can help us understand "structuralism" better in general, including its categorical versions (as will become clearer later).

2. The Varieties of Mathematical Structuralism: Extending the Taxonomy

As the discussion in the previous section shows, it is misleading to speak of "structuralism" as if this label attached to a unique, unified position in the philosophy of mathematics. (Occasionally "structuralism" is identified, even more misleadingly, with Shapiro's position, since it is the most prominent one.) Rather, a whole variety of "structuralist" positions have been proposed in the literature. They all share the core idea with which we started this introduction, namely that "mathematical theories characterize abstract structures." But how that slogan is interpreted varies widely. Previously we used a threefold taxonomy so as to introduce some order and clarity. It started with Parsons's distinction between eliminative and non-eliminative forms of structuralism, with Hellman's and Shapiro's positions as paradigm cases. Then we added categorical structuralism as a third alternative, one that is not easy to compare to the others. But actually, the options one should consider are more varied than that; hence, a comprehensive taxonomy for structuralism has to be broader and richer. We will now take some steps in that direction.

One further distinction (largely ignored for long, but related to the difficulties in comparing categorical and other forms of structuralism) is very basic and should be introduced before all the others. It is the distinction between "methodological structuralism," on the one hand, and "metaphysical structuralism," on the other (cf. Awodey 1996; Reck and Price 2000). In related terminology, one can distinguish between "mathematical" and "philosophical structuralism." In what follows, we will treat these two dichotomies as the same (with only a slight difference in what is highlighted). The former term in each case, i.e., "methodological/mathematical structuralism," is meant to capture a distinctive way of doing mathematics, i.e., a certain "methodology," "form of practice," or "mathematical style." (For related discussions, cf. Corry 2004, Carter 2008, and Landry 2018, among others.) Roughly, it consists of doing mathematics by "studying abstract structures"; but this slogan requires again clarification. In addition, the methodology at issue comes with a general assumption on what mathematics is about, or what its subject matter is, namely "abstract structures." Then again, methodological/mathematical structuralism does not include, in itself, claims about what these structures are, i.e., about their "nature," "abstractness," "existence," etc. That is exactly what is added when we move on to "metaphysical/philosophical structuralism." In other words, there is a basic distinction between one kind of structuralism focused on "methodological," or more generally "mathematical," issues, while the other kind adds specific "metaphysical," or more broadly "philosophical," theses to the mix. Hence the labels.

With respect to mathematical practice, or to pursuing mathematical research fruitfully, one typically does not need to consider the specific "metaphysical" or more generally "philosophical" questions just mentioned. In fact, mathematicians often dismiss them as misleading or misguided (with important exceptions, as we will see). In contrast, it is exactly such questions that philosophers of mathematics try to address, including Benacerraf, Resnik, Shapiro, Hellman, and Parsons. Of course, the philosophers' answers should be grounded in mathematical practice, i.e., the goal should be a philosophical position not only compatible with but informed by mathematical practice, thus appropriate for it. Categorical structuralists often try to remain on the methodological/mathematical side alone. Their concern is then how to think through, and develop further, the methodology emerging in the late 19th- and early 20th-century mathematics by category-theoretic means. But sometimes metaphysical/philosophical views are added along the way also along such lines (e.g., when category theory is interpreted in a formalist way).

One main goal of the present collection of essays is to clarify the origins, and with it the nature, of methodological/mathematical structuralism up to the rise of category theory (from Grassmann, Dedekind, and Klein to Noether, Bourbaki, and Mac Lane). This is intended to clarify what it has meant, and still often means, to do mathematics by "studying abstract structures." A second main goal is to illustrate that the emergence of methodological/mathematical structuralism, in that sense, was accompanied, from early on, by reflections that shade over into "metaphysical/philosophical structuralism." And it was not only philosophers who engaged in such reflections, but also mathematicians themselves. (In several cases, the philosopher and mathematician at issue is one and the same person.) To be able to pursue this second goal fruitfully, several further distinctions concerning "structuralism in mathematics" are called for, now especially on the philosophical side.

Parsons's dichotomy between eliminative and non-eliminative forms of structuralism will remain helpful in what follows, so that we will keep using it. But it should be added, right away, that one can find relevant positions in the literature that are "semi-eliminativist," unlike Hellman's position, which is "fully eliminativist." This concerns structuralist positions that reject the postulation of structures as distinctive, independent abstract objects, but accept other kinds of abstract objects, e.g., sets (thought of in some nonstructuralist way then). In other words, there are structuralist positions that are eliminativist about structures, but are not nominalist. They still count as forms of eliminative structuralism, but not of eliminativism about abstract objects generally.

One example is what is sometimes called "set-theoretic structuralism" (cf. Reck and Price 2000). According to this position, the natural numbers, say, should not be identified, in any strict or absolute sense, with the finite von Neumann ordinals. Why not? Because, exactly as Benacerraf argued, there are various set-theoretic models of the Dedekind-Peano axioms, indeed infinitely many, and none of them is privileged in a metaphysical sense (as opposed to some weaker pragmatic sense). This becomes a form of structuralism if one adds that "any set-theoretic model will do," so that the intrinsic, nonstructural properties of its elements do not matter. In other words, we can identify "the natural numbers" with the finite von Neumann ordinals, but do so in a pragmatic sense and with the proviso that we could have identified them with, say, the finite Zermelo ordinals too. In John Burgess's words (Burgess 2015), this position involves a "indifference to identify" them with any particular model of the Dedekind-Peano axioms; similarly in other cases. (Strictly speaking, this position is "structuralist" with respect to some objects but not generally, e.g., not for sets.)

One can generalize this approach. Set-theoretic structuralism is a specific version of "relativist structuralism" (see again Reck and Price 2000). This name derives from the fact that the reference of "the natural numbers," and with it the reference of the numerals "1", "2", "3", etc., is relative to an arbitrary, or only pragmatically determined, choice between equivalent models. Other forms of relativist structuralism result then from modifying the basic framework. For example, one can work not just with pure sets, but also allow for "atoms" or "urelements." Along such lines, one can, in fact, let any objects whatsoever occupy any "position" in a given structure; thus Julius Caesar or some beer mug can "be" the number 2. (If at least some abstract objects are included as candidates here, this will again be a semi-eliminative view broadly speaking, but also a form of eliminative structuralism.)

Yet another kind of structuralism, closely related to relativist structuralism, is "universalist structuralism" (cf. Reck and Price 2000). With it, we come back to if-then-ism, i.e., the suggestion that any mathematical sentence should be seen as quantifying over all models of the relevant axiom system and as consisting of a corresponding if-then claim. In other words, we keep the "universalist" side of Hellman's position but leave out its modal aspect. But what about the existence of the models; i.e., what about the so-called non-vacuity problem for the theory at issue? Or what ensures its "coherence"? Here one can again work with axiomatic set theory as the framework; but there are other options as well. (Once more, this makes the position partly but not fully "structuralist.")

Turning to the side of non-eliminative structuralism, there are additional options available too and further distinctions to be drawn. (At this point, we go beyond Reck and Price 2000.) One of them is indicated, implicitly, by Shapiro's label "*ante rem* structuralism." Shapiro's terminology, explicitly inspired by medieval debates about universals, suggests "*in re* structuralism" as an alternative. (Another alternative might be *post rem* structuralism. Parsons's position has been labeled that way, although this terminology is not widespread. We will not pursue it further here.) In fact, two different forms of *in re* structuralism have played a role in the literature already. For the first, consider again the natural numbers within a set-theoretic framework. There are infinitely many models for the Dedekind-Peano axioms, as we have noted. But then, we can identify "the structure" of the natural numbers with the equivalence class (under isomorphism) of all of them. This class is different from all the models in it, while arguably depending on them ontologically (the way in which a class depends on its elements). In that sense, we have arrived at a form of *in re* structuralism. Actually, this is exactly the position one gets if Russell's "principle of abstraction" (cf. Russell [1903] 1996) is applied to the case at hand, as Rudolf Carnap and others noted.

As the appeal to Russell's "principle of abstraction" indicates but as is true more generally, there are certain forms of structuralism that arise from "structuralist abstraction" (cf. Schiemer and Wigglesworth 2018; Reck 2018). That abstraction can, in turn, be reconstructed as a mathematical function, which maps models of a mathematical theory to a corresponding "abstract structure" as their value. Along Russellian and Carnapian lines, that value is the class of all models isomorphic to the given one (or more generally, equivalent in some other way). A different option is to use the following "abstraction function": it maps any given model of a theory to a novel, privileged model of it. (In the case of the Dedekind-Peano axioms, say, the value then deserves to be called "the natural numbers"; similarly for "the real numbers," etc.) Here the new model is again ontologically dependent on the original ones, since it has been introduced "by abstraction" on their basis. In the recent literature, this position has been explored by Øystein

Linnebo and Richard Pettigrew, building on Dedekind. For these authors, the "principle of abstraction" involved is similar to neo-Fregean "abstraction principles" (cf. Linnebo and Pettigrew 2014; Reck 2018).

Last but not least, let us return to eliminative structuralism once more. Yet another option under that label, different from Hellman's and Chihara's, is "concept structuralism," as advocated recently by Dan Isaacson, Solomon Feferman, Tony Martin, and others (cf. Isaacson 2010; Feferman 2014). The guiding idea for them is that what matters in mathematics in the end is "concepts" as opposed to "objects." Thus, there is the concept "model of the Dedekind-Peano axioms" (in Russell's terminology: "progression"; in Dedekind's: "simple infinity"); likewise for other axiom systems that define (higher-order) concepts, including the concept of set. All that is crucial for mathematics, so the suggestion now, is what is provable from those concepts, thus what is true for all models falling under them. Once again, we avoid postulating "abstract structures" as separate objects. We might even say that the structure simply "is" the concept at issue, parallel to its identification with the (closely related) equivalence class given earlier, except that the structure is not an object in this case.

3. The Pre-History: Key Themes and Features

As should be evident by now, a plethora of positions have been introduced under the name of "structuralism in mathematics" since the 1960s, following the initial lead of Benacerraf, Putnam, later Resnik, Shapiro, Hellman, Parsons, and others. For the most part, they are versions of "metaphysical/philosophical" structuralism." But these positions are all inspired by mathematical practice, at least implicitly, thus by methodological/mathematical structuralism. So far we have not said much about what the latter amounts to, except for mentioning category theory as one version, or one outgrowth, of it. However, it is not the only version, much less the original one. To probe this issue in a deeper way, it becomes important, and will prove illuminating, to consider how "structuralist mathematics" arose historically since the middle of the 19th century. Many of the essays in the present volume will, in fact, address that rise in detail, i.e., they are meant to fill that gap. As further preparation for them, we will now offer a brief overview of the themes and features that play a key role.

A number of developments transformed mathematics radically in the 19th century, as is now widely acknowledged, so much so that some commentators have talked about a "second birth" of the discipline (Stein 1988). The result of that transformation was "modern mathematics." In the 20th century, it was then systematized, provided with a set-theoretic foundation, and later reshaped, once again, along category-theoretic lines. The main innovations that played a role in the 19th century are well known (see, e.g., Boyer and Merzbach 1991, chaps. 24–26). They include the radical broadening and rethinking of geometry, by means of introducing various non-Euclidean theories (projective, elliptic and hyperbolic, *n*-dimensional, etc.); the rigorization and arithmetization of analysis, including better and more explicit characterizations of the number systems involved (from the natural to the complex numbers), and leading to a broadened conception of function as well; the transformation of algebra, from the study of equations to a much more general, abstract conception of it (Galois theory, the introduction of novel number systems and related innovations, e.g., quaternions, vector spaces, etc.); and the rise of set theory and modern logic (transfinite numbers, generalized notions of set and function, quantification theory, and a logical theory of relations, among others).

These broad developments brought with them several important changes that we consider to be "proto-structuralist," i.e., part of the immediate background for the rise of "structuralist mathematics" but not constitutive of it yet. They include: the rejection of the traditional view that mathematics is "the science of number and quantity," by adding parts that cannot be understood thus (complex analysis, group theory, topology, etc.); the expansion and systematization of traditional theories, by introducing "ideal elements" (points at infinity, points with complex coordinates, ideal divisors, transfinite numbers, etc.); later the reconstruction of such objects in set-theoretic terms (Dedekind cuts and ideals, quotient constructions in algebra, etc.); the adoption of the view that many parts of mathematics are not about particular objects and their properties, but are applicable much more widely (group theory, ring theory, topology, etc.); the related suggestion that mathematics is more about the relations between objects than about their intrinsic, non-relational properties (from number systems to groups, rings, etc.); also the emphasis on the "freedom" of mathematics, in the sense that its development should not be constrained by its direct and readily apparent applicability, but should involve the exploration of new "conceptual possibility" (non-Euclidean geometries, transfinite numbers, etc.); and finally, the suggestion that many parts of mathematics, perhaps even all, can be reconstructed systematically within "logic," including a basic theory of sets and functions (thus basing it on "laws of thought" alone).

As the reader will see, many of these changes play important roles in the essays in this volume. In fact, one function of these essays is to document their increasing significance in 19th- and early 20th-century mathematics. But the features we have listed also brought with them, or soon led to, additional innovations that are more properly "structuralist." Prominent among those are the following six, as we want to suggest: First, there is the suggestion to base various parts of mathematics on fundamental, characteristic concepts ("group," "field," "metric space," also "simple infinity," "complete ordered field," "3-dimensional Euclidean space," etc.); and this leads to the modern axiomatic approach (explicitly in Peano, Hilbert, etc.). Second, the relevant concepts typically specify global or "structural" properties (the "denseness" of an ordered system, the "continuity" of a space, also the "infinity" of a set); and this relies on considering whole systems of objects, as opposed to individual objects, especially various "complete infinities" (the systems of the real numbers, Euclidean space, various function spaces, etc.). Third, increasingly important becomes the study of such systems by relating them to each other, especially in terms of morphisms (homomorphisms, isomorphisms, etc.). A case in point, but also a method applicable more generally, is, fourth, the characterization of various systems or kinds of objects via "invariants" (complex-valued functions via their Riemann surfaces, geometries via their groups of transformation, etc.). Fifth, there is the novel practice of "identifying" isomorphic systems, since they are "essentially the same" from a mathematical point of view (e.g., different models of geometric theories, the system of Dedekind cuts and that of equivalence classes of Cauchy sequences, etc.). Sixth, this can all be seen as culminating in the view that what really matters in mathematics is the "structure" captured axiomatically, on the one hand, and preserved under relevant morphisms, on the other hand (two closely related techniques, both important historically).

What makes a mathematical methodology structuralist, in our view, is not the presence of one or two particular items on the list just given; nor do all six have to be present. Rather, what matters is the self-conscious and fruitful use of several of them together. Put differently, we think it is neither promising nor appropriate to try to define structuralist mathematics in terms of a few essential features (necessary and sufficient conditions). Instead, what we are dealing with is a case of "family resemblances," and hence, of "clusters" of these features emerging and playing a central role. In any case, when all of the corresponding tools and techniques were in place, in the late 19th and early 20th centuries, mathematicians began to study the results more systematically. This led to the introduction of several additional fields in mathematics: axiomatic set theory, seen as a "foundation" for all of mathematics (not just as an exploration of the infinite, although this too remained a goal); model theory, proof theory, recursion theory, etc., as ways to study "metamathematical" or "metalogical" features of mathematical theories (consistency, completeness, and categoricity, but also decidability, mutual interpretability, etc.); and somewhat later, category theory, with its generalization of the use of morphisms, invariants, etc. (initially in algebra and topology, then also more widely, and finally as an alternative foundation for mathematics).

During the period when these innovations became accepted widely, a number of philosophically inclined mathematicians and mathematically informed philosophers also began to reflect on their deeper significance, often in conversation with each other. For many of them this included attempts to say more about how to conceive of the nature of the various "structures" that had arisen, or of the underlying notion of "structure." This means, as we will see, that already toward the end of the 19th and early in the 20th century one can find forms of metaphysical/philosophical structuralism in the literature. And as we would like to emphasize, this happened 60–80 years before Benacerraf, Putnam, etc., began to publish on the topic, i.e., long before what is usually seen as the start of the debates about the topic. A central goal of the present collection is both to recover and to make fruitful this "prehistory of mathematical structuralism".

4. Previews of the Essays, Indicating Their Contributions to the Volume

After this condensed survey of structuralist themes and key features that arose in 19th and early 20th century mathematics, the stage is set for the essays in this volume. In this section of the introduction, we will preview the main themes in them, thus also indicating how each of these essays fits into the volume as a whole.

Overall, the volume is divided into two parts. The essays in Part I are concerned primarily with aspects of methodological/mathematical structuralism as they emerged in the 19th and early 20th centuries. Each focuses on a particular mathematician, from Grassmann to Mac Lane. With Part II, the focus shifts to the metaphysical/philosophical side, as well as to contributions by philosophers. However, the division between the two parts is porous, including many cross-references in the essays themselves. Moreover, while most of the essays in Part II focus on figures usually identified as philosophers, such as Peirce, Russell, and Cassirer, some of the people covered in this part, like Poincaré and Bernays, were also mathematicians, perhaps even primarily so. Why are the essays on them then included in Part II? The reason is that the main focus of these essays is on philosophical (and logical) themes. Yet even by that criterion, some placements of essays could have been different.

Among mathematicians, Richard Dedekind is often regarded as the "founding father" of structuralism; second in that regard is David Hilbert (cf. Shapiro 1996); and third probably Nicolas Bourbaki, especially among historians of mathematics (cf. Corry 2004). All three will be quite prominent in our volume, but it reaches back further, thus starting with Grassmann.

More precisely, the volume starts with an essay by Paola Cantú on Hermann Grassmann, the author of *Die Lineare Ausdehungslehre*, a book that influenced various later structuralists strongly. As Cantú documents, Grassmann suggested conceiving of mathematics as a "general theory of forms," and this was related to his introduction of several new systems of "quantities" (hyperspaces, hyperreal numbers, etc.). In fact, Grassmann emerges as an early proponent of concept structuralism, thus of eliminative structuralism.

In contrast, Dedekind has been interpreted as the first "non-eliminative structuralist" in the literature (Reck 2003), although this is not uncontroversial. In the essay co-written by Erich H. Reck and José Ferreirós in the present volume, the main focus is instead on Dedekind's contributions to methodological/mathematical structuralism. That essay starts with an account of important influences on Dedekind, namely Gauss, Dirichlet, and Riemann. Then his structuralist contributions to algebra and algebraic number theory, including Galois theory, are discussed, making evident their close relation to his work on the foundations of arithmetic and set theory.

A mathematician usually not associated with structuralism, nor recognized much as a philosopher of mathematics more generally, is Moritz Pasch. In Dirk Schlimm's essay, Pasch's work, not only on geometry but also on arithmetic, is put in the context of broader developments in 19th-century mathematics. In doing so, structuralist features of his approach are revealed, e.g., concerning the centrality of duality principles, even though a tension remains with the empiricism that dominates his work philosophically. In addition, Schlimm provides an analysis of what should be seen as central to mathematical/methodological structuralism more generally.

In the next essay, by Georg Schiemer, the investigation of 19th-century geometry with respect to the rise of mathematical structuralism is continued. Here it is Felix Klein's use of group theory in reconceptualizing geometry that becomes the focus. Klein was led to rethink the subject matter of different kinds of geometry in terms of what is invariant under relevant groups of transformations. This culminated in his "Erlangen program," in which various geometries are classified by comparing their respective transformation groups. Another influential structuralist idea one can find in Klein, as Schiemer documents, is the suggestion to show the structural equivalence of different geometries in terms of "transfer principles".

In Wilfried Sieg's essay on David Hilbert, two kinds, or uses, of the axiomatic method are distinguished: there is "structural axiomatics," on the one hand, which grew out of Hilbert's early axiomatization of geometry; and there is "formal axiomatics," on the other hand, which involves the metamathematical study of axiomatic systems in Hilbert's later proof theory. With respect to the former, the "conceptual" methodology advocated earlier by Dedekind and others is brought to full fruition, i.e., their suggestion to base various parts of mathematics on "characteristic concepts." The latter constitutes a major, and very influential, example of studying mathematical theories with respect to "foundational" issues, such as consistency and decidability, by using tools from modern logic.

Another mathematician in the early 20th century who built on Dedekind's work explicitly was Emmy Noether. In Audrey Yap's essay, three phases in Noether's mathematical career are distinguished. It is especially the second and third phases that are relevant for our purposes, since they illustrate the shift from a more concrete, calculational way of doing mathematics, still dominant in Noether's first phase, to a more and more abstract approach. Moreover, the latter became a paradigm of methodological/mathematical structuralism later in the 20th century, strongly influencing the work of Bourbaki and the rise of category theory, among other developments.

The name "Nicolas Bourbaki" stands for a group of mathematicians who worked on reshaping and systematizing modern mathematics from the 1930s on, by building on what they found in Dedekind, Hilbert, Noether, and others. According to Gerhard Heinzmann and Jean Petitot's essay, what lies at the core of the methodology that resulted is a "functional conception of structure." Its main purpose was to help mathematicians in reconceptualizing the interrelations of different theories and, especially, in solving hard problems. The latter is illustrated by an extended case study from algebraic geometry, which leads us from Dedekind through André Weil to Alain Connes. Here issues concerning methodological/mathematical structuralism are illustrated by means of a substantive mathematical example, one that still occupies mathematicians today.

Both in the essays on Noether and Bourbaki, and also already in the essay on Klein, close connections between 19th- and early 20th-century mathematics, on the one hand, and category theory, on the other, start to emerge. This theme is deepened in Colin McLarty's essay on Saunders Mac Lane. In that essay, Mac Lane is presented as a mathematician interested in logical and philosophical issues from early on, although he became disillusioned by their treatment in mainstream philosophy. Later he was led back to some of them from within mathematics. As a result Mac Lane adopted, and promoted explicitly, a form of methodological/mathematical structuralism tied to category theory. McLarty characterizes it as "a working theory of structures for mathematicians."

The first essay in Part II of our volume concerns the logician, philosopher, and scientist C. S. Peirce. (Like Part I, this second part is arranged chronologically by the birthdates of the thinkers under discussion.) In the recent literature, Peirce has been interpreted as subscribing to a form of non-eliminative structuralism (Hookway 2010). In Jessica Carter's essay, the focus is instead on Peirce's distinctive, still relatively unknown views about mathematical inquiry and proof, namely in terms of diagrammatic reasoning. Carter finds some aspects of structuralism in Peirce's works, at least in the sense of methodological/mathematical structuralism. But she refrains from interpreting him as a full-fledged structuralist, since this would oversimplify his multifaceted work.

The second essay in Part II, by Janet Folina, concerns Henri Poincaré. This essay, in particular, could have been put into Part I too, since Poincaré made major contributions to structuralism as a mathematician. But Folina is more interested in metaphysical/philosophical ideas and themes, which one can find in Poincaré's writings as well. She argues, in particular, that Poincaré should be seen as a proponent of *ante rem* structuralism. However, in this case one needs to separate the *ante rem* aspect clearly from the realist aspect, as she adds, even though they are often conflated in the current literature on structuralism. In fact, with respect to mathematics Poincaré turns out to be a "constructivist *ante rem* structuralist," surprising as that may sound at first.

As a prototypical logicist, Bertrand Russell tends to be seen as a strong opponent of structuralism. There is justice to this view, although the story is more complicated and more interesting in the end, as Jeremy Heis documents in his essay. Early in his career, during the years 1900–1903, Russell was intensely interested in Dedekind's works, as some of his posthumously published writings show; and he interpreted Dedekind as holding a non-eliminative structuralist position. While attracted to that position initially himself, he then turned against it, for reasons Heis documents in detail. But Russell was an important contributor to the debates about structuralism in another way as well, namely by means of his promotion of a logic of relations. That logic was taken as the background for reconstructing structuralist ideas by several later thinkers, from Ernst Cassirer in the early 20th century to Geoffrey Hellman today.

Cassirer's explicit and detailed defense of structuralism, both in the methodological/mathematical and in the metaphysical/philosophical senses, is the topic of Erich Reck's second essay in this volume. While Cassirer was very knowledgeable about Felix Klein's work and about developments in 19th-century geometry more generally, the focus in this essay is on his positive reception of Dedekind's structuralist views. This included a defense of them against Russellian objections. But Dedekind's contributions are also embedded into a rich account of the history of mathematical science, guided, among others, by Cassirer's distinction between "substance concepts" and "function concepts".

The last three essays in the volume concern Paul Bernays, Rudolf Carnap, and W. V. O. Quine, respectively. In Wilfried Sieg's second contribution, an essay on Bernays, the connection between methodological/mathematical structuralism, in Dedekind's, Hilbert's, and related works, and 20th-century proof theory is thematized. The core concept for Sieg is that of a "methodological frame," as introduced in Bernays's writings. The role of such frames is to allow for a kind of "reductive structuralism," in the sense of investigating mathematical theories in terms of their underlying deductive structures, thus by utilizing the tools of Hilbertian proof theory. Seen as such, Bernays's work constitutes a reflection on mathematical structuralism from the perspective of mathematical logic.

In Georg Schiemer's second contribution to this volume, Rudolf Carnap's work from the 1920s–1930s is investigated in a parallel way, i.e., with respect to its use of logic. As Schiemer documents, Carnap picked up on Russell's "principle of abstraction," both to demystify the notion of "structure" and to study it further logically. This led him to a form of *in re* structuralism according to which "the structure" of a mathematical theory, such as Dedekind-Peano arithmetic, is identified with the equivalence class of models that satisfy the theory.

In the last essay in our volume, Sean Morris discusses Quine's place in the prehistory of structuralism. Several current structuralists, including Resnik, Shapiro, and Parsons, have acknowledged Quine as a strong influence. Usually this involves Quine's later works, in which he proposed a very general form of structuralism (also for physical objects, not just for mathematical ones). However, Quine's relevant views can be traced back to his earliest publications and his dissertation, as Morris documents. He also argues that Quine stands firmly in the tradition of Russell's and Carnap's "scientific philosophy," including the rejection of traditional metaphysics. Hence Quine's structuralism should not be seen as exemplifying any strong form of realism. Instead, it is grounded in the methodology of the mathematical sciences as interpreted by him.

In fact, the latter holds, *mutatis mutandis*, also for every other figure covered in the present volume. What the essays establish as a whole, then, is that there are very strong ties between methodological/mathematical forms of structuralism and more metaphysical/philosophical views. These should not, and ultimately cannot, be understood independently of each other; they are two sides of the same coin.

5. Gaps in this Volume and Two Final Suggestions

While this collection of essays is meant to recover the prehistory of mathematical structuralism in a substantive and inclusive way, we realize that it is far from complete. In other words, we could, and perhaps should, have included a number of other thinkers and developments as well, both on the mathematical and on the philosophical sides.

One mathematician who comes to mind right away is Bernhard Riemann. Riemann is mentioned in several of our essays, but he would undoubtedly have deserved his own treatment. A second, less prominent example is Hermann Hankel. He too comes up in some of our essays along the way, with his view of mathematics as a "theory of forms" that is similar to Grassmann's. A third example is George Boole, as well as other British algebraist in the mid-19th century, who helped to push mathematics in a structuralist direction too.

Actually, some ideas relevant for us can already be found in late 18th- and early 19th-century thinkers. Abel and Gauss are two cases, with their suggestion that mathematics is more about relations, relations of relations, etc., than about objects. And a few such ideas can be traced back even further, e.g., to D'Alembert's work on the calculus (cf. Folina's essay), or to Leibniz's study of the continuity of space (cf. De Risi in progress). But the further back one goes, the more one should speak of "proto-structuralist" rather than "structuralist" ideas, as we believe.

On the side of philosophers there are gaps too. One notable figure mentioned only tangentially, but who would have deserved a separate essay, is Edmund Husserl. As is well known, Husserl started out as a mathematician, including by serving as an assistant of Weierstrass in Berlin. And he was concerned about a "general theory of manifolds" in some of his later works, thereby building explicitly on Grassmann's, Riemann's, and Klein's writings. There are clear connections to methodological/mathematical structuralism in his works; but one can find related metaphysical/philosophical views too, including perhaps another form of *in re* structuralism.

Somewhat later in the 20th century, another interesting philosopher for our purposes is Albert Lautman. While still largely unknown among English-speaking philosophers, he offered detailed reflections on the mathematics of Bourbaki, and with it, on mathematical structuralism. Lautman's views were mathematically and philosophically sophisticated, thus deserving to be reconsidered. Indeed, we had planned to include essays on both Husserl and Lautman; but because of space and time restrictions, they had to be omitted in the end. And beyond Husserl and Lautman, there surely are further philosophers one could have included. Then again, the volume is already very long as it is.

Because of such omissions, the volume is open to complaints that we did not cover this or that figure who would undoubtedly have deserved a separate essay as well. In response, we want to close with two suggestions: First, one thing we hope this volume will do is to inspire more research on the prehistory of structuralism, thus recovering and reinvestigating other relevant mathematicians and philosophers as well. In other words, we suggest viewing the volume only as the start with respect to covering its topic. Having said that, we hope that it is substantive enough to inspire further work.

Second, while the approaches and treatments in our essays are primarily historical, we hope that the volume will be seen as a contribution to mathematical structuralism in a systematic sense too, i.e., as relevant for current philosophy of mathematics. As we see it, combining historical and systematic investigations can only enrich both sides, also in other cases. More generally, a rich topic such as mathematical structuralism will surely benefit from being studied in several different ways.

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