2

Grassmann's Concept Structuralism

Paola Cantù

1. Introduction

It is hard to determine whether Hermann Grassmann should be considered a mathematically inclined philosopher or a philosophically inclined mathematician, for he was an autodidact in mathematics (he learned mathematics mainly from the books of his father, Justus, and from Legendre's treatise), in Greek philology, and partly also in philosophy. He studied theology at the Berlin University at the end of the 1820s, and attended, among the philosophy courses, only Schleiermacher's lectures on dialectics and Ritter's lectures on the history of philosophy. In any case, his main contributions concern mathematics and linguistics, rather than philosophy. Or rather, he got recognition mainly for his mathematical results and his linguistic achievements, whereas his philosophy of mathematics did not receive similar attention, not even after his death. Yet a large part of Grassmann's mathematical work is specifically devoted to (*a*) the relation between the emergence of a new abstract mathematical theory and the need for a new philosophical frame to understand it, (*b*) the relation between certain applications of this theory and Leibniz's universal characteristics, and (*c*) the characterization of mathematical disciplines by means of a philosophical deduction of their fundamental concepts and of mathematics as the science of particulars generated from a given element.

Notwithstanding the growing number of publications concerning specific aspects of Grassmann's mathematical or philosophical writings,[1] it is still difficult to find a comprehensive treatment of his philosophy of mathematics. There are several reasons for this: (1) Grassmann's philosophy of mathematics varies in different writings, (2) it is difficult to clearly distinguish his conception from that of his brother Robert, (3) where a distinction can be traced, Robert appears to have been the one who was most interested in logic and philosophy of logic

[1] See, for example, Banks (2013); Radu (2013); Petsche et al. (2011); Schubring (2005); Flament (2005); Radu (2003); Darrigol (2003); Schubring (1996a); Dorier (1995); Schreiber (1995); Flament (1994); Boi et al. (1992); Châtelet (1992); Otte (1989); Hestenes (1986); Schlote (1985); Echeverría (1979); Lewis (1977); Heath (1917).

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(see Peckhaus 2011 and Grattan-Guinness 2011), (4) Grassmann's philosophical style, typical of early 19th-century natural philosophy, cannot easily be read by contemporary philosophers, (5) Grassmann was interpreted in different ways in the second half of the 19th century and at the beginning of the 20th century (see, e.g., Hankel 1867; Cassirer 1910; and Klein 1875). These interpretations exemplify Grassmann's philosophical destiny, which is perhaps less "tragic" than his mathematical fortune, yet not really fortunate, because he was often *used* to corroborate a given conception of mathematics rather than *read* to verify what his own view really was. These interpretations did not adequately emphasize the role of particulars in Grassmann's mathematics (Cassirer), the role of intuition and the reasons for a quasi-axiomatic presentation of extension theory and arithmetic (Klein), the differences between the general theory of forms and a symbolic treatment of mathematical objects as signs whose referent does not matter (Hankel). Yet all these aspects are extremely relevant to grasp Grassmann's understanding of concept formation in mathematics and his contribution to the history of methodological and philosophical structuralism. I will try to reconstruct Grassmann's definition of mathematics as the science of the particular, and to investigate his complex distinction between formal and real, referring to some philosophical interpretations discussed in Lewis (1977), Flament (1994), Banks (2013), and Schlote (1996).

So in the following it will not be sufficient to recall several of Grassmann's mathematical contributions that are relevant for the structuralist transformation of mathematics, such as abstract algebra, linear algebra, and number theory (§2). The most important task will be that of giving a plausible and comprehensive reconstruction of Grassmann's philosophy of mathematics (§3), as it emerges from his own mathematical works, rather than from subsequent influential interpretations, such as those by Hankel, Cassirer, and Klein. As a result, it will emerge that the notions of linear combination, series, and addition are more important to Grassmann than the notions of function, mapping, and order. Mathematics is the science of the particular, and the general theory of forms does not properly belong to it, because it is about underdetermined connections.

The main aim of the chapter will be to analyze Grassmann's contribution to structuralism, discussing differences and similarities between our interpretation and some received views in the literature (§4). In particular, I will try to evaluate Grassmann's work with respect to two different issues that are often mixed up in the literature or, when they are clearly distinguished, are often called by different names or defined in slightly different ways: methodological (or mathematical)[2] and philosophical structuralism.

[2] In the literature, this methodology is often called "mathematical structuralism" rather than "methodological structuralism." I prefer Reck and Price's (2000) terminological choice for two reasons. On the one hand, this choice does better justice to the idea that the structuralist philosophical

By *methodological structuralism* I intend an analysis of the method that is applied by mathematicians when they are doing mathematics and that has evolved in time. Reck and Price have defined methodological structuralism as a methodology that "motivates, explicitly or implicitly, many of the structuralist views in the philosophical literature" (2000, 345). Reck and Schiemer in the introduction to this volume enucleate a list of conditions that should characterize methodological structuralism. Later in this chapter, I broadly follow their suggestion and associate methodological structuralism with questions concerning (1) criticism of mathematics as the science of a given domain of objects (e.g., quantities), concerning objects in isolation rather than relations, (2) the role of intuition and formal deductions, (3) the role of axioms, invariants, and applications, and (4) the relation between alternative ways to frame mathematics (e.g., set theory, category theory). This methodological structuralism tackles deep philosophical questions, which often arise in mathematical practice itself or in historical analysis of the development of mathematical theories.

Philosophical structuralism is used here as a collective name for a large number of different philosophical theories centering on the fundamental question, "What is a structure?" Typical issues concern, for example (1) whether there are objects and operations, and what their relations to structures might be, (2) whether general structures can be distinguished from particular structures and from exemplars, (3) what is that we call "formal" in a structure and what role is played by axiomatics within it. In section 4.3 the analysis of these issues is interconnected with the study of answers given in the contemporary philosophical debate by Shapiro, Parsons, Feferman, Isaacson, and Burgess. A tentative distinction between concept structuralism and object structuralism is used to characterize Grassmann's own perspective with respect to some contemporary approaches.

The objective is certainly not to determine whether Grassmann was a forerunner of a specific philosophical position in the contemporary debate. This would be quite anachronistic, because both mathematics and philosophy have deeply evolved from Grassmann's time. On the one hand several conceptions of structuralism are grounded either in a set-theoretic or in a categorical framework that had not yet been developed at the time; on the other hand the analytic approach to structuralism is based on a new understanding of mathematics and logic introduced, e.g., by Dedekind, Frege, Peano, Russell, and Hilbert, which makes it difficult to separate our common use of certain notions (such as

viewpoint emerges in mathematical practice, and that a study of the mathematical method might already be philosophical in nature. On the other hand, it avoids the mistake of considering the methodological component of structuralism as the only mathematical aspect of it, whereas also the so-called philosophical structuralism might be the result of mathematical self-reflection.

function, concept, equality) from the corresponding use made by Grassmann. Yet, provided that historical differences are spelled out clearly, it is not anachronistic to evaluate Grassmann from the perspective of contemporary philosophy of mathematics, to verify whether he asked questions that challenge certain structuralist views or raised issues that still need to be clarified.

2. Grassmann's Mathematics

Hermann Grassmann's contributions to mathematics and to its applications to physics are numerous; we will recall them very shortly. A clear and detailed presentation of Grassmann's mathematical writings can be found in Schubring (1996b) and Petsche et al. (2011). In the following we will restrict our attention to several contributions that might be relevant for the development of structuralism and that derive mainly from the following works: *Ausdehnungslehre* (both in the 1844 and in the 1862 revised edition), *Geometrische Analyse, Lehrbuch der Arithmetik*, and Robert Grassmann's *Formenlehre*.

2.1. Linear Algebra

Grassmann's extension theory (ET) (*Ausdehnungslehre*) introduces several fundamental concepts of linear algebra: basis, dimension, generator, linear dependence and independence, but there is no axiomatization of the theory (Dorier 1995; Zaddach 1994). Grassmann's vector theory is developed in a purely abstract way (in modern parlance, the vector system is a module over a field), and conceptually distinguished from geometry, which is considered as an applied science (it is the application of ET to three-dimensional space).

Grassmann's theory partially differs from contemporary vector-based systems, such as vector analysis, exterior algebra, and geometric algebra, both from a technical and from a philosophical point of view. Differences concern the closure of the operations, the condition of homogeneity on addition, and the conception of the product (Cantù 2011, 96–98). Besides, an important characteristic of Grassmann's system is that his notions of base and of a system of (independent) generators does not aim at the introduction of a system of coordinates, but rather at expressing the idea that all the magnitudes of the system are characterized by some generating law.

Following Grassmann, who uses a geometrical analogy to make the abstract presentation more intuitive, we will introduce the fundamental notions of ET (element, generating law, simple extensive formation, extensive magnitude) by analogy with geometry (point, movement, bound vector, vector). An extensive formation (*Ausdehnungsgebilde*) is "the collection of all elements into which the generating element is transformed by continuous evolution":[3] geometrically speaking, it is the geometrical figure resulting from the different positions of a point in continuous movement. An elementary (*einfach*) extensive formation "is produced by continuation of the same fundamental evolution" (Grassmann 1844, 48, my trans.): geometrically, it is a straight line that results from the movement of a point in just one direction.

An extensive magnitude is the class of extensive formations that are generated according to the same law by means of equal evolutions (Grassmann 1844, 48–49); that is, the vector defined as an equivalence class of bound vectors having the same direction, the same orientation, and the same size.

Given Grassmann's understanding of equality as an identity whose criterion is substitutivity, one cannot say that two extensive formations (two bound vectors) are equal (in the sense that they are equivalent), but rather that their extensive magnitudes (their corresponding free vectors) are equal (see §2.3.1). An extensive formation is determined by the elements it is composed of. An extensive magnitude, on the contrary, is determined only by direction, size, and orientation; that is, it does not depend on the initial element of the generation (Grassmann 1844, 49).

2.2. Number Theory

2.2.1. Natural Numbers

The theory of natural numbers is presented by Grassmann in the *Lehrbuch der Arithmetik* (1861), which is the result of collaboration with his brother Robert. Here the term "magnitude" (*Grösse*) replaces "form"; mathematics is defined as the science of magnitudes, that is, of anything that should be set equal or unequal to another thing (Grassmann 1861, 1). This general definition of magnitude might apply to any kind of form: arithmetical, extensive, or combinatorial. In any case, arithmetical magnitudes are characterized by a further property, that is, the fact that they are obtained by successive applications of a specific kind of connection (an addition) to a single magnitude taken as given and denoted by the sign *e*. It should be noted that Grassmann does not mention the number 1 as the arithmetic unit. Any magnitude that is taken as initial element to build the arithmetic series, which he calls *Grundreihe*, by successive addition of that initial magnitude

[3] See Grassmann (1844, 48; 1995, 47). Cf. also Lewis (1977, 150). By translating *Ausdehnungsgebilde* by "extensive formation" rather than "extensive structure" (Kannenberg) or "extensive entity" (Lewis), I follow here the French translation by Flament and Bekemeier in Grassmann (1994).

can play the role of a unit. The commutativity and associativity of any arithmetical magnitudes denoted by the symbols *a, b, c* is not introduced as an axiom, but derived inductively (*inductorisch*) from the commutativity and associativity of $a + e = e + a$ and $(a + b) + e = a + (b + e)$ respectively (Grassmann 1861, 1). This shows the essential role played by the initial element and by the operation of addition in the definition of an arithmetical magnitude, and thus of the notion of series (see §3.1.1).[4]

The *Lehrbuch* has been very influential, because it introduces (1) a clear distinction between the symbols used to denote (*bezeichnen*) the concepts and the concepts themselves, (2) the parallelism between the symbolic development (*Formelentwicklung*) and the conceptual development (*Begriffsentwicklung*) of a proof, (3) a clear separation between primitive and derived propositions, and (4) the use of induction as a method of inference.

2.2.2. Real Numbers

Real numbers are not introduced in arithmetic, but in ET. Grassmann, at least in the first edition of the *Ausdehnungslehre*, defines real numbers as ratios of extensive magnitudes of the same dimension: they are thus introduced as magnitudes of grade zero, that is, as magnitudes that have no dimension. The idea that numbers are themselves magnitudes is familiar in modern linear algebra, where the real number field can itself be represented as a vector system (a module on the field of real numbers). In particular, the fact of having no dimension allows for the product of real numbers to be commutative, even if the product between extensive forms is generally non-commutative. So all properties of the usual arithmetical operations hold for the so-introduced real numbers, which are the only magnitudes whose product commutes (Cantù 2011, 98).

Once real numbers have been introduced according to the operation that generates them (division), they can be used as a tool in the symbolic definition of extensive magnitudes given in the second edition of the *Ausdehnungslehre*. Relying on an analogy with the generation of natural numbers as successive additions of the unity, Grassmann defines several unit magnitudes $e_1, e_2, \ldots$ and then introduces extensive magnitudes as additions of the products of these units by real numbers, as in the following polynomial: $a_1e_1 + a_2e_2 + \ldots$. Yet real numbers, although presupposed in the definition, can still be conceived as extensive magnitudes “if the system consists only of the absolute unity (1) (Grassmann 1862, 12, my trans.).[5]

[4] The notion of power series emerges already in arithmetic, because Grassmann investigates which powers can be transformed into a power series of the form $ax^n + bx^{n-1} + cx^{n-2} + \ldots$ with x as base. The complex relations between the solution of systems of linear equations, analysis, and extension theory is here evident.

[5] This sentence is omitted in Grassmann (2000). Grassmann thus uses the notion of series to express natural numbers, extensive magnitudes, and also real numbers. Besides, he often uses it as a tool

Natural numbers are based on addition of absolute unities, rational numbers are based on division of natural numbers, and real numbers are obtained as the quotient of extensive magnitudes. This approach does not provide a unified notion of number that includes natural, rational, and real. Grassmann does not seem to be bothered by the piecemeal character of the definition. On the contrary, he aims to ground each kind of number in the operation that is used to generate it, and seems to consider as most primitive those notions that are built on the basis of addition alone (see §2.3.1 for a discussion of this algebraic hierarchy between operations). So natural numbers are more primitive than rational numbers because the former are introduced by an operation of addition, whereas the latter need multiplication and division. For the same reason, extensive magnitudes are more primitive than reals, which are magnitudes and not numbers: extensive magnitudes are introduced by addition, whereas reals are obtained as the quotient of extensive magnitudes.

2.3. Algebra and Logic

2.3.1. Abstract Algebra

Under the name of "general theory of forms" (GTF) Grassmann gathers the investigation of equality, difference, and the common properties of some connections that make their appearance in all branches of mathematics. Contrary to the usual treatment in modern algebra, he does not investigate sets of objects endowed with a given operation, but rather considers the connections in a purely formal way, abstracting from the elements they might be applied to. It is true that sometimes he reasons as if in specific mathematical branches one should consider the connections as always holding between certain given magnitudes, and then show that these connections satisfy the requirements that allow one to call them addition and multiplication respectively (Schlote 1996, 168). Yet this can be done only once GTF has been established.[6] This explains why Grassmann claims that

in the solution of problems in different mathematical branches. The ubiquity of the notion of series as well as its capacity to express the generating rule of mathematical forms attests to the foundational role Grassmann attributes to it.

[6] See, e.g., the following passage in the second edition of the *Ausdehnungslehre*: "We therefore also call such a method of conjunction a multiplication, provided only that its multiplicative relation to addition is demonstrated, or in other words, provided only that the equal entry of all the terms of the conjunctive factors into the conjunction is established in the above sense" (Grassmann 2000, 43). In other words, Grassmann defines *in abstracto* what addition, multiplication, and raising to a power are, and then, given a domain closed under an operation, he determines whether it is an addition, a multiplication, or a raise to a power, and specifies its further characteristics.

GTF should precede all other mathematical branches in the exposition: there are both epistemological and didactic grounds, because GTF provides a foundation of all other branches of mathematics—in that it presents as united what should be united, and has the highest degree of generality—and also because it spares useless repetitions of basic concepts in a mathematical treatise (Grassmann 1844, 28).

Two forms are said to be equal when they can be substituted one for the other in any connection they occur in. Equality is transitive—if two forms are equal to a third, then they are equal one to the other—and has the following property: forms that are generated in the same way from equal forms are equals (Grassmann 1844, 28).

Forms are determined by their generating law, and are therefore equal if the same law from the same initial element generates them. Grassmann has often been criticized for his adoption of a Leibnizian conception of equality as substitutability *salva veritate* instead of a Euclidean conception of equality as equivalence (Helmholtz 1887, 377n): as I read him, his equality lies midway between Leibniz and Euclid, because he defines it as an identity, and restricts it to some features of concepts rather than defining it between objects themselves.[7]

Given that forms are not given objects but the results of an act of thought that generates them according to a certain law, only the characteristics that depend on the specific way in which forms have been generated will be taken into account in the comparison: the substitutability is thus limited to pertinent contexts.

Grassmann then considers three connections and introduces a four-level distinction based on their decreasing generality.

1. Grassmann believes that the most restrictive conditions to be required from any mathematical connection depend on the number of connections that are introduced and on their reciprocal relations. So he requires from a first connection (*connection of first order*) that it be commutative and associative, and from a second connection related to the first (*connection of second order*) that it satisfy the distributive laws with respect to the first.[8] At this level of generality, the two connections (denoted respectively by $\cap$ and $\cup$) are pre-mathematical operations between concepts.
2. Then there is the formal level, where the conditions are less restrictive and the connections of first and second order are respectively called "formal addition" and "formal multiplication" and denoted by the usual arithmetical symbols $+$ and $\cdot$. A formal addition is a simple synthetic connection with

[7] E.g., two vectors can be considered as equal—in some extended sense—because their directions and lengths are equal, i.e., identical (Grassmann 1844, 28).

[8] The distributive laws are two, because the second connection need not be commutative.

a single-valued analytical operation, whereas a formal multiplication is a connection of second order with respect to the given addition. This is the level of GTF, which is occasioned by an investigation of certain properties of the connections that are common to different mathematical branches.

In modern parlance, one could say that Grassmann's notion of formal addition corresponds to a commutative group, and the notion of formal multiplication corresponds to a ring under two operations (Schlote 1996, 168). Yet it has often been remarked in the literature (e.g., in Lewis 1977, 140, 146, and Flament 1992, 216) that one should not consider the properties of the connections of first and second order, or the properties of the formal addition and formal multiplication as axioms,[9] or as a reductionist kind of foundation. I believe that a comparison with ancient proportion theory might be illuminating, because—as Aristotle himself observed—the theorems of the theory of proportions could be demonstrated not only separately for numbers and for geometrical magnitudes but also in a more general way. Just as the formulation of proportion theory did not imply (at least not until the 18th century) the creation of a new genus of objects (Cantù 2008, §3), the fact that Grassmann assembled a list of propositions that "relate to all branches of mathematics in the same way" (Grassmann 1844, 33; 1995, 33) does not imply the construction of a new branch of mathematics or of a new domain of objects. So formal operations have the properties that are common to real operations, the latter being the operations that generate the mathematical forms in each mathematical discipline. This explains why Grassmann considers addition as being always commutative: he had not encountered any example of a non-commutative additive group in mathematics.

3. Third, there are abstract connections between thought forms, which might have different properties depending on the thought forms they are applied to. For example, at this level we find addition and multiplication between natural numbers, or addition and multiplication between extensive magnitudes. These abstract connections might be different (e.g., the multiplication is commutative in the first case and not commutative in the second case), but only with respect to the properties that were not already contained in their respective "formal" notions. These are what Grassmann calls "real" connections between forms: they are "real" because the law of connection is specified and grounds the generation of the forms.[10] This is

[9] See Radu (2003) for a discordant point of view on this issue.

[10] "So far we have developed the concept of addition in a purely formal manner, since we have defined it from the validity of certain laws of conjunction. This formal concept also remains the only general one. Yet it is not the way we arrive at this concept in the individual branches of mathematics. Rather in them a characteristic method of conjunction is obtained from the generation of the magnitudes itself, which manifests itself as an addition in precisely the general sense given,

the mathematical treatment of connections, as it is developed in its mathematical branch. The idea that in each branch of mathematics, one should verify whether the connections that can be introduced can be called addition or multiplication confirms that there should be a distinction between the level of formal operations and the level of specific mathematical branches, where Grassmann refuses to admit a domain of elements given prior to, or independently from, the generation of the elements themselves (Cantù 2011, 100).[11]

4. Fourth, there is the application of mathematical operations to physical reality, as in the case of the addition of masses or forces, or segments. To this level belongs the investigation of the connections that one finds in geometry.

Mathematics, as we will see in section 3.1, is for Grassmann the science of the particular. GTF, on the contrary, investigates formal operations, which are necessarily underdetermined, because the nature of the forms and their generating law induce the properties of the operation, which might vary relative to the domain of application. Grassmann considers as more "general" the product relative to a variable domain—a domain that is not closed under the operation but rather a result of our carrying out the operation itself. It is more general in the sense that it is underdetermined, because the determination or particularization of the operation depends on further conditions dictated by the nature of mathematical objects and by generating rules. The refusal to admit a domain of elements given prior to, or independently from, the generation of the elements themselves is an idea that Grassmann never abandons, and a basic assumption of his "constructivism" (see §4.2.2 and Cantù 2011, 100).

To resume, Grassmann's GTF, that is, the study of some fundamental relations and operations that occur in all branches of mathematics, is not itself part of mathematics, because it contains formal operations that are underdetermined and that might receive full determination only when they become real operations in mathematical disciplines or in the applications of mathematics.

since those formal laws apply to it" (Grassmann 1844, 40; 1995, 39, trans. slightly modified). "We have therefore formally defined the general concept of this multiplication as well; if the nature of the magnitudes to be conjoined is given, then to this formal concept must correspond a real concept that expresses the method of generation of the product by the factor." (Grassmann 1844, 44; 1995, 42, trans. slightly modified).

[11] It is interesting to note that different notions of product might occur in the same mathematical branch: for example, in the case of ET, there are a real addition between homogenous magnitudes (e.g., between segments) and a formal addition between non-homogeneous magnitudes (e.g., the addition between a point and a segment gives a point, because the symbol of addition has to be interpreted as a movement from one point to another point rather than as a concatenation of magnitudes) (Cantù 2011, 97).

2.3.2. Logic

Hermann Grassmann's contributions to logic concern some reflections (1) on the notion of primitive proposition—and in particular on the idea that in formal sciences there are only definitions, and no axioms, because mathematics concerns abstract concepts and not given objects—and (2) on a general logical law (*law of progression*).

The nature of primitive propositions varies according to the kind of science in question: formal sciences start from definitions, while real sciences start from axioms.

> Now if truth is in general based on the correspondence of the thought with the existent, then in particular in the formal sciences it is based on the correspondence of the second thought process with that existent established by the first, that is, on the correspondence of the two thought processes. . . . Consequently, the formal sciences cannot begin with axioms [*Grundsätze*], as do the real; rather, definitions comprise their foundation. (Grassmann 1844, 22; 1995, 23, trans. slightly modified)

Unlike Kant's formal criterion of truth, which concerns only the form and not the content of knowledge, and thus cannot say anything on the eventual contradiction between knowledge and its object (Kant 1787, 82), Grassmann's condition on the consistency of two thought processes is a condition on the consistency between an object of thought (the result of the first thought process) and a thought, that is, between two concepts.

Grassmann mentions a law of progression (*Fortschreitungsgesetz*) that he considers a general logical law:[12] it guarantees that "anything that is asserted about a series of things in the sense that it should hold for each individual of the series can actually be asserted about each individual belonging to the series" (Grassmann 1844, 65, my trans.).[13]

[12] Here "logical law" should be intended as a law of thought, rather than as a law of propositional or predicate logic. This interpretation is supported by what Grassmann's brother Robert explicitly claimed, i.e., that mathematical proof is independent of any given natural language, and of any logical theory (syllogistic logic in particular). Even if Robert's conception differs from that of Hermann inasmuch as he treated GTF as being itself mathematics, and logic as being one of its branches (Grassmann 1872, 17–18), I think the two brothers would agree that the notion of series is more pervasive in science (and thus more fundamental or more general) than universal instantiation in predicate logic: Hermann does not mention the latter at all, neither in GTF nor in mathematical branches such as arithmetic and extension theory, whereas Robert considers universal instantiation as occurring in a specific mathematical branch: logic.

[13] On the rule of progression cf. Cassirer (1910, 20–21). I translate "law of progression" rather than "procedural principle," as does Kannenberg (Grassmann 1995, 62), to underline the relationship to the concept used by Cassirer in a passage of *Substanzbegriff und Funktionsbegriff* where he discusses Grassmann's ideas ("In all these cases, we are not concerned in analyzing a given 'whole' into parts similar to it, or in compounding it again out of these, but the general problem is to combine any conditions of progression in a series in general into a unified result" [Cassirer 1910, 127; 1923, 96]).

This law, which we could understand as universal instantiation for series (rather than a law explaining the universal quantification of a predicate), just makes explicit what Grassmann means by general proposition, and should not be assumed as an axiom of mathematics. It is rather what allows us to demonstrate mathematical results, because proofs are considered concatenations (*Aneinanderketten*) of definitions, that is, themselves series of thoughts.

3. Grassmann's Philosophy of Mathematics

3.1. Mathematics as the Science of the Particular

Defining mathematics as the science of (thought) forms, Grassmann claims that mathematics is about concepts, which are considered particulars generated by means of an act from some initial element.

1. Thought forms are particulars that have "come to be through thought" (Grassmann 1844, Intro., §§2–3, 22–23).
2. Thought forms might come to be by different types of generation and different ways to relate them to the initial element of the generation (Grassmann 1844, Intro., §5, 25).

But he also characterizes mathematics by contrasting its peculiar method to the method followed in philosophy.

3. The mathematical method goes from the particular to the general (Grassmann 1844, Intro., §13, 30).

According to both characterizations of mathematics, which are not mutually exclusive but rather complementary, mathematics is conceived as the science of the particular.

3.1.1. Mathematical Thought Forms as Particulars

Mathematical thought forms (*Denkformen*)—or simply forms—are determined by their generating law: any form is a particular being that has come to be by some act of thought (it is the result of a particular act).

> Pure mathematics is therefore the science of the *particular* existent that *has come to be* by thought. The particular existent, viewed in this sense, we call a thought form or simply a *form*: thus pure mathematics is the *theory of forms*. (Grassmann 1995, 24)

Table 1 The Partition of Mathematics

Elements	Discrete generation	Continuous generation
Equal	Arithmetic (natural numbers)	Analysis (intensive magnitudes)
Different	Combinatorics (permutations)	Extension theory (extensive magnitudes)

Forms are abstract concepts that result from a generating thought from an initial element, which is itself a particular concept. Mathematics is thus the science of the particular that is posited by thought and not the science of the general laws of thinking (logic).[14]

The generation of the forms is so intrinsic to their nature that it also explains the partition of mathematical disciplines: depending on the relation between the elements (equal or different) and on the kind of generating law (discrete or continuous) that is applied to an initial element, one obtains a partition of mathematics into four branches: arithmetic, analysis, combinatorics, and extension Theory (see Table 1).

The partition of mathematics is based on the generating law and on the relation of the generated element to the initial element; that is, it is based on operations and relations, but also on the determinateness of the initial element. One reason why Grassmann introduced the term "form" in the definition of mathematics is that it contains a reference to formation, that is, to the way mathematical particulars are generated by a certain law, which alone guarantees their determinateness.[15]

> Since what is different from something given [*von einem Gegebenen*] may be different in infinite ways, the difference would get lost completely in the indeterminate, were it not constrained by a fixed law. (Grassmann 1995, 29, trans. modified)

[14] "The formal sciences treat either the general laws of thought or the particular as established by means of thought, the former being the dialectic (logic), the latter pure mathematics" (Grassmann 1995, 23–24).

[15] Other reasons might be the influence of Leibniz as well as dissatisfaction with the usual term "magnitude" (§4.2.1).

Grassmann is a conceptualist and a constructivist: what mathematics is about are concepts determined by construction, according to a law, from a particular that is considered as given (even if it is itself a concept).[16]

The relation of difference and equality between elements is introduced as a difference or equality with something given, that is, a first element from which the forms are successively generated. The importance of the individuation of a first element in the series becomes particularly relevant in arithmetic, where it grounds Grassmann's conception of numerical induction, but it plays a fundamental role also in ET, where it grounds the notion of generator, and in applications (e.g., in the geometrical calculus the initial element is the point, whereas in the barycentric calculus it is the point magnitude). Yet this characteristic of Grassmann's philosophy of mathematics has often been neglected or underestimated in the literature.

However, it plays an essential role in the notion of equality (e.g., two extensive magnitudes are equal if they are generated in the same way by equal elements), in the definition of thought form, and in the clarification of the specificity of the mathematical method (§3.1.2). Besides, it is relevant in the distinction between operations that are defined on a fixed domain and operations whose domains depend on the forms they are applied to: see, for example, the regressive or applied product (Grassmann 1844, 206), which is relative to the system that two magnitudes have in common (Cantù 2011, 94). Finally, it constitutes an essential aspect of Grassmann's understanding of concept formation, which is better represented by the notion of series than by the notion of function. Whereas the notion of function is introduced in modern mathematics by a correspondence between two given sets of elements, a series is always determined by an initial element and a law of development.[17]

One of the most acute interpreters of Grassmann's notion of series was Ernst Cassirer. Yet he underestimated the role of the initial element that is taken as given in order to go from the particular to the general, as well as the additive group of elements required by the notion of a series, insisting rather on what he calls the "concrete universality" of mathematical functions, and on the order relation between the members of a series. Cassirer understood Grassmann's claims about the initial element in ET by analogy with arithmetic rather than with geometry, and tried to go beyond the limit to "definite kinds of transformations" by highlighting the fact that "the element . . . is . . . only a pure particular grasped as different from other particulars," whereby no "specific content" is assumed

[16] By concept I mean what Grassmann calls "a thought representing an existent"; the existent might be given independently from thought, as in real sciences, or be itself a thought, as in formal sciences.

[17] In modern parlance we could say that a function could be intended as a logical notion, whereas a series is usually considered a mathematical notion.

(Cassirer 1923, 97–98).[18] But Grassmann explicitly adds that a further determination and distinction is guaranteed by the generation law, which might generate the forms in a discrete or in a continuous way. This is needed in order to let a real concept of operation correspond to the formal concept of operation.

Cassirer rightly underlined what he considered the "most general function of the mathematical concept: . . . giving some qualitatively definite and unitary rule that determines the form of the transition between the members of a series" (1923, 98). But then he concluded by inscribing Grassmann among the authors who considered mathematics as the science of relations:

> The considerations by which Grassmann introduces his work thus create a general logical schema under which the various forms of calculus, which have evolved independently of the *Ausdehnungslehre*, can also be subsumed; for they only show from a new angle that the real elements of mathematical calculus are not magnitudes but relations. (Cassirer 1923, 99).

So, even if Cassirer's reading of Grassmann is partially accurate and faithful, he inscribes Grassmann in a tradition to which Grassmann does not properly belong, especially if one acknowledges that Grassmann's GTF, which actually deals only with relations and operations, does not really belong to mathematics.[19]

[18] "First, in place of the point, that is, of the particular position (locus), we here substitute the *element*, by which we mean a mere particular, conceived as distinguished from other particulars; and indeed we attribute to the element in the abstract science absolutely no other content. There can therefore be no question as to what sort of particular it properly is—for it is the particular *per se*, devoid of any real content—, or in what sense this one is distinguished from the others—for it is merely defined as the distinct *per se*, without establishing any real content that might account for its distinctness. Our science has this concept of an element in common with combination theory, and thus the designation of elements by different letters is also common to both. The difference consists only in the way forms are obtained from the elements in the two sciences; that is, in combination theory by simple conjunction and thus discretely, but here by continuous generation. The different elements can now also be regarded as different states of the same generating element, and this difference of stages corresponds to differences in position (locus)" (Grassmann 1844, 47; 1995, 46, trans. modifified). Cf. also Grassmann (1994, 12).

[19] Erich Reck rightly noticed that Cassirer's notion of function cannot be reduced to the concept of a mapping between two domains. I agree that the notion of series grounds what Cassirer says about conceptual understanding: an intuitive multiplicity can be understood conceptually only if its elements can be seen as the elements of a series (Cassirer 1910, 19–20). Yet the notion of function is distinct from the notion of series: it is "some general law of arrangement through which a thoroughgoing rule of succession is established. . . . it is the rule of progression, which remains the same, no matter in which member it is represented" (Cassirer 1910, 20–21; 1923, 17). In another passage, Cassirer again distinguishes between "a series which has a first member, and for which a certain law of progress has been established, of such a sort that to every member there belongs an immediate successor with which it is connected by an unambiguous transitive and asymmetrical relation, that remains throughout the whole series" and the "progression" (*Fortschritt*) in which "we have already grasped the real fundamental type with which arithmetic is concerned" (Cassirer 1910, 49; 1923, 38). On the one hand, the notion of function is influenced by Dedekind's notion of *Abbildung* (Cassirer 1910, 50); on the other hand, Cassirer apparently agrees with Russell that arithmetic is a formal study of relations (Cassirer 1910, 48).

More specifically, Cassirer's move from series to relations is too quick. Grassmann's philosophy of mathematics is rooted in the notions of series and additive group rather than in the notions of function and order: the former and not the latter are relevant in concept formation.

In the introduction to the second edition of the *Ausdehnungslehre*, Grassmann explains the different role played by the notion of linear combination, which is essential to defining the notion of elementary extensive magnitude, and by the notion of function, which is not an expression between signs: its value is itself a magnitude, and precisely a composite magnitude (Grassmann 1862, 7; 2000, xvi, trans. modified). The notion of function occurs only in the second part of the book and presupposes the notion of magnitude, which is defined by means of the notion of linear combination.

> Definition. When a magnitude *u* depends on one or several magnitudes *x, y*, . . . in such a way that, whenever *x, y*, . . . assume determinate values, then *u* also assumes a determinate (univocal) value, then we call *u* a function of *z, y*, ... (Grassmann 1862, 224, my trans.)

So, according to Grassmann, neither linear combinations nor functions are notational abbreviations (Grassmann 1862, 5; 2000, xiv): they are rather essential tools for concept formation. It is only because a new autonomous concept has been formed by addition and multiplication that inverse operations can arise and the concept of a negative quantity can be introduced. In particular, Grassmann's notion of function has nothing in common with the notion of a mapping between two domains, because the latter does not include any privileged reference to operations. And even in the case of numbers, the operation of successor is considered a generating law that builds numbers by addition (the simplest example of linear combination, characterized by a single unit: the absolute unit). Thus, Grassmann highlights the similarities between numbers and extensive magnitudes, which are generated by similar concept formation tools, rather than their respective differences (the kind of order, and the commutative property of the product).

3.1.2. Mathematical (and Scientific) Method: From the Particular to the General (and Back)

The notion of a particular is at the basis of the mathematical method too. Mathematics and philosophy are characterized by an opposite movement: philosophy starts from the general and arrives at the particular with an analytical process of decomposition of a complex concept in more simple concepts; mathematics proceeds in a synthetic way and links several particulars to get a new particular, that is, links several concepts to get a new concept. The philosophical (dialectical/logical) method and the mathematical method can be better understood by analogy with the two different moments of the Platonic dialectic process: reduction and division. Philosophy starts from an overview of the totality, which is successively articulated and ramified, whereas mathematics starts by connecting particulars and aims at their unity (Grassmann 1844, 30).[20]

It is still controversial whether Schleiermacher's *Dialektik* had a decisive influence on Grassmann's understanding of mathematics.[21] Without pretending to give historical support to the claim, I would like to recall two issues of Schleiermacher's *Dialectic* that might clarify my interpretation of the difference between formal and real operations as a difference in determinateness, and offer a key to understanding the importance of the initial element in Grassmann's generation of mathematical objects.[22]

Schleiermacher distinguishes between the construction of the one (the initial element) and the combination of the one with another one (generation law starting from an initial element and determining the other elements). Even if mathematics mainly deals with combination, it does not ignore that each particular is in turn the result of a thought process, and in particular cannot ignore that the initial element is the result of a construction, which is relevant in the determination of the outcome of the combination. Production (or construction) and combination condition each other reciprocally (Schleiermacher 1839 [1986], 179).

The knowledge of a single concept obtained by a process of concept formation is an incomplete knowledge that needs to be further determined by the connection of that concept with other concepts. The mathematical method concerns specifically the determination of connections, but these cannot be separated from the particulars they should connect, and from the initial element whose knowledge should be completed by further determinations.[23] The initial element

[20] For a convincing interpretation of Schleiermacher's influence on Grassmann's distinction between dialectic and mathematics see Lewis (1977). See especially the distinction between construction (concept formation) and combination (connection of particular concepts).

[21] See, for example, Engel (1911), Lewis (1977), and Petsche (2004), who claim it did, and Schubring (2008), who denies it—or at least restricts the influence to other domains than mathematics, as, for example, philology.

[22] Both issues are actually mentioned in Lewis (1977), but are not related to Grassmann's *Ausdehnungslehre*, nor specifically to Grassmann's ideas on concept formation and indeterminacy. Schleiermacher's influence is rather recognized by Lewis in Grassmann's deduction of mathematical branches by opposition of the fundamental concepts (equal, unequal, discrete, continuous). See also Schubring (2008, 63). I thank Jamie Tappenden for calling my attention to this review.

[23] "Even if a concept is formed it can never by itself completely represent the existent since as concept it only contains what is based in the particularity of this existent and not what is posited in it as a consequence of its associations [with other existents]. . . . Thus each given thought contains the requirement of finding another new thought and of determining that which is left undetermined. The first is the extensive direction within combination and the latter the intensive, and it will be in the oscillation between the two that we must progress. The method in the first direction—to find from a given knowledge a new one—we call the heuristic; that in the other—to connect the dispersed

of the series allows the construction of the successive element, which in turn adds determinateness to the preceding element.

The scientific method, which is for Grassmann the method of presentation of a science in a treatise, that is, a pedagogical method (Flament 1992, 215), should sum up in itself both the specificity of the philosophical method (which proceeds from the general to the particular) and the specificity of the mathematical method (which proceeds from the particular to the general). So any scientific exposition should combine two aspects, which Grassmann calls respectively content and form: the content (*Inhalt*) of a science is the development that goes from one individual truth to another individual truth in demonstrations, whereas the form (*Form*) is a guiding idea, which is either a presentiment of the searched truth or a conjectural analogy with other well-known branches of knowledge (Grassmann 1844, 31).

3.2. Formal and Real

There is an ambiguous use of the terms "formal," "form," and *Formel* in Grassmann, which explain the difficulty in understanding and appreciating his philosophy of mathematics. These expressions occur in different contexts with different meanings.

1. Forms are thought forms, which are opposed to what exists independently of thought ("das Sein als das dem Denken selbständig gegenübertretende"), to what is given and cannot be itself generated by thought (e.g., space) (Grassmann 1844, Intro., §1, p. 22). The former notion is connected to the distinction between "formal sciences" and "real sciences," whereby formal sciences concern the purely conceptual (*rein begrifflich*), whereas real sciences concern what is given outside thought (e.g., spatial [*räumlicher Natur*] notions) (Grassmann 1844, 22).
2. *Formeln* are symbolic expressions, as opposed to their denotations: concepts (Grassmann 1861, 6).
3. "Formal operations" in GTF are the formal addition and the formal multiplication that are opposed to "real operations" (e.g., addition between numbers in arithmetic or between extensive magnitudes in ET) (Grassmann 1844, 41, 42n).
4. The form of a science is its treatment or exposition, as opposed to its content (concatenations of truths) (Grassmann 1844, 31).

and isolated given material—the architectonic" (Schleiermacher [1839] 1986, 179–180; trans. in Lewis 1977).

In the following, I will discuss in more detail the first three occurrences, having already discussed the last one in the method of scientific exposition (§3.1.2).

3.2.1. Purely Conceptual versus Spatially Intuitive

We have already mentioned that thought forms are the objects of mathematics. "Formal" is also used to characterize several sciences in contrast to real sciences: the main differences between them concern their respective objects (abstract vs. intuitive), their primitive propositions (definitions vs. axioms), and their criterion of truth (correspondence between two acts of thought versus correspondence with some external thing). Independence from intuition is clearly stated at the end of Grassmann's *Geometrische Analyse*, where abstract as purely conceptual is opposed to real as associated with spatial intuition.

> Now, in fact, as is demonstrated throughout Grassmann's *Ausdehnungslehre*, all concepts and laws of the new analysis can be developed completely independently of spatial intuition [*unabhängig von der räumlichen Anschauung*], since they can be tied only to the abstract concept of a continuous transformation; and, once one has grasped the idea of this purely conceptually conceived [*rein begrifflich gefassten*] continuous transformation, it is easy to see that also the laws developed in this essay can be conceived as freed from spatial intuition [*von der räumlichen Anschauung gelösten*]. (Grassmann 1995, 384, trans. modified)

The main point is not to do away with intuition, but to give it its own role. The analogy with geometry is essential in the method of exposition of the *Ausdehnungslehre* and as a heuristic guide in the search for theorems. One thing is to consider pure concepts as independent from intuition and another thing is to assume that Grassmann calculates with signs devoid of meaning.

3.2.2. Symbolic versus Conceptual

Formel occurs in expressions such as *Formelentwicklung*, where it might be considered synonymous with what we now call symbolic. In the *Treatise on Arithmetic*, formal as symbolic is mentioned in the inferential development of arithmetical truths, which are expressed by symbols but denote concepts. There is thus no idea of symbols or "signs or letters the referent of which did not matter" in Grassmann (Darrigol 2003, 522). This formalistic interpretation of Grassmann's number theory has been encouraged by Hankel, who maintained a distinction between actual and formal numbers, but identified formal numbers with signs (Hankel 1867, 36). Quoting Grassmann as a fundamental source of his work (Hankel 1867, 16), Hankel indirectly suggested that Grassmann shared his point of view. Yet Grassmann claimed that

> "the symbolic development [*Formelentwicklung*] and the conceptual development [*Begriffsentwicklung*] should go hand in hand. . . . The whole treatment will proceed along a conceptual development, whereas the formulas added at each step symbolically represent the conceptual advancement." (1861, 6, my trans.)

The mentioned definition of function, as well as the refusal to consider linear combination as a notational abbreviation, also supports a non-formalistic reading of Grassmann.[24]

3.2.3. Formal versus Real: Two Levels of Abstractness

The adjective *formell* occurs in GTF as a way to distinguish formal addition and multiplication from real addition and multiplication. Here the formal concerns an underdetermination of the concept of a connection, which gets embodied and becomes real only in each specific mathematical discipline. Both the formal connections and the real connections are thus abstract and purely conceptual (*rein begrifflich*), and thus opposed to intuitive or spatial notions that can be found in applied mathematics (e.g., in a real science as geometry). The ambiguity of the terminology is here evident and explains why it is difficult to understand Grassmann's philosophy of mathematics. The notion of real connection, opposed to that of formal connection, is not to be found in real sciences, but in formal sciences! It is thus abstract and opposed as such to what is real in the sense of concrete, as connections between geometrical figures.

Are the formal operations of GTF merely expressed by signs devoid of reference, as several authors took them to be? Here Grassmann's idealistic philosophy explains why this is not the case.

> As the general sign for conjunction we take the symbol $\cap$; now if a and b are the factors, with a the prefactor, b the postfactor, then we indicate the product of their conjunction as $(a \cap b)$, where the parentheses here express that the conjunction indicates that the factors are no longer separate, but that their concepts are unified. (Grassmann 1995, 34)

It is certainly true that the level of generality and the abstraction from specific features of the real operations suggest that the formal connections do not refer in the same way, because they concur in the formation of concepts once applied

[24] For a different reading see Darrigol (2003, 522), but also Klein, who encouraged a formalistic reading of Grassmann, as he praised his ingenious algorithms (Klein 1979, 166–167). Klein's reading suggests that ET contains algorithms that refer to geometry, thereby ignoring Grassmann's abstract level that is situated between the formal and the real concrete level (§2.4).

to some concept. Just as proportion theory (from which GTF inherits the analysis of equality and of addition) needs to be applied to specific mathematical branches, so does Grassmann's GTF. The following passage supports this interpretation, according to which formal and real should be understood as disembodied and embodied respectively:

> Incidentally, it lies in the nature of things that the conceptual determination of these connections is here purely formal, whereas only in the single sciences it can be embodied by means of real definitions. (Grassmann 1844, 42n, my trans.)

The insistence on the separation of the different mathematical branches and on the purity of proofs in each domain is incompatible both with the idea that GTF has as its object an abstract structure, and with the view that it constitutes a symbolic calculus devoid of reference or meaning.

3.2.4. Hankel's Three-Level Distinction

Hankel rightly distinguishes the first and the second notions of formal and real, individuating three levels in the *Ausdehnungslehre*: formal, real abstract, and real concrete, which correspond to the laws of GTF, of extensive magnitudes and of geometrical figures respectively (Hankel 1867, 16–17).

Hankel thus traces a distinction between (1) the level of formal laws, (2) the level of abstract content, and (3) the level of real content. There are at least two distinct interpretations of this tripartition in the literature on Grassmann: (*a*) universal algebra, different algebras, and physical instantiations of such algebras, (*b*) abstract algebra, linear algebra, and geometry. A critical discussion of these two interpretations is useful to understand Grassmann's contributions to universal algebra, abstract algebra, and non-Euclidean geometry, and therefore to the transformation of mathematics into a science of structures, but it is also useful to compare contemporary philosophical structuralism with Grassmann's peculiar understanding of mathematical objects and structures.

The second interpretation of the tripartition holds for what Grassmann does in the *Ausdehnungslehre*, provided that one also remarks that of the three levels, only the second properly pertains to pure mathematics, whereas the former is not mathematics, and the latter is applied mathematics, and provided that one recalls the differences between Grassmann's approach and modern algebra. The main question here is whether GTF (1) does not belong to mathematics yet, because it has not been sufficiently developed or because it cannot be part of mathematics, given that it is only formally abstract and not really abstract, or (2) cannot belong to mathematics, given its too general nature. In support of interpretation (1), it should be noted that Grassmann himself declares that "such a general branch is not yet available" and that he has developed it only as far as it is needed for ET, thereby neglecting a third possible connection: raising to a power (Grassmann 1844, 33, 42n). In support of interpretation (2) there is the fact that, according to Grassmann's conception of the mathematical method, GTF cannot belong to mathematics, because it does not go from the particular to the general. And in fact the connections are considered independently from their application to a first element.

My claim is that GTF should be considered as something that has to do with the scientific method, which, as we have seen, has to incorporate both the philosophical and the mathematical method, going both from the unity of the idea to the multiplicity of particulars and back. This interpretation seems to be confirmed by the role Robert Grassmann assigns to it in the *Formenlehre*. Robert develops what Grassmann calls GTF as a theory of magnitudes (*Grösenlehre*),[25] including the general definitions and theorems that make a rigorous scientific thought possible, teaching us how to make scientific inferences (*wissenschaftlich beweisen*) (Grassmann 1872, 1), and characterizes it as the general part in opposition to special disciplines.[26]

Now, if one considers not only the *Ausdehnungslehre* but more generally the totality of Grassmann's writings, then one might have an argument for the first interpretation of the tripartition already mentioned. The level of formal laws might correspond to a certain way of doing universal algebra; the level of abstract content might correspond to different algebras developed by Grassmann, among which is vector space theory, but also some non-commutative algebras that he developed in his essay on different kinds of multiplication (Grassmann 1855); and the level of real content would correspond to geometry, to the barycentric calculus, and to other physical instantiations of such algebras.

[25] Yet Robert Grassmann uses a different terminology and inverts the presentation: he begins by raising to a power (*Anreihung*, which is not commutative), then introduces multiplication (*Einigung*, which is associative), and finally introduces addition (*Vertauschung*, which is commutative) (Grassmann 1872, 15–24). He then considers direct (*Trennung* or *trennbare Knüpfung*) and inverse operations (*Lösung* or *untrennbare Knüpfung*), where the former are univocal and the latter are not univocal.

[26] "*Grösenlehre*, the first or most general discipline of *Formenlehre*, teaches us to recognize those connections between magnitudes that are common to all disciplines of *Formenlehre*. It develops the laws of equality, addition or *Fügung*, multiplication or *Webung*, and exponentiation or *Höchung*. The four special disciplines of *Formenlehre* emerge from *Grösenlehre* through the introduction of new conditions" (Grassmann 1872, 11–12, my trans.).

4. Is Grassmann a Structuralist?

4.1. Mathematical Contributions

Grassmann's contributions to mathematics already tell us something about his relation to methodological structuralism (see §1). If the mathematical structuralist methodology is the result of several important innovations such as abstract algebra, axiomatic method, set theory, and Bourbaki's structuralism (Reck and Price 2000, 346), Grassmann did explicitly contribute to the first factor, thanks to his contribution to vector space theory, which clearly favored the development of abstract algebra. Vector space theory is also an example of a non-Euclidean geometry, because the vector space has dimension n, with $n \in \mathrm{N}$, and thus includes the investigations of abstract spaces with dimension > 3. Grassmann's distinction between real and formal sciences thus contributed to the liberation of abstract geometry (linear algebra) from physical space.

Even if Grassmann's presentation of extensive forms is not strictly axiomatic, Grassmann contributed to the development of axiomatics for at least three reasons:

1. He gave an axiomatic presentation of natural numbers in 1861, where, thanks also to the collaboration with his brother Robert, specific attention was given not only to the propositions chosen as axioms but also to demonstrative inferences (and in particular to which propositions are used in each step of the derivation).
2. He developed a purely abstract treatment of linear magnitudes that is completely independent from concrete intuition.
3. He was the source of inspiration of Giuseppe Peano, who published an explicitly axiomatic presentation of vector theory in 1888 and of arithmetic in 1889.[27]

Grassmann contributed to the investigation of the abstract structure of a system of extensive magnitudes. He highlighted similarities and differences concerning the operations of different systems of mathematical forms. On top of that, he favored a comparison of the abstract structures of numbers with the abstract structures of magnitudes, individuating their main difference in dimensionality and in the commutativity of the product. Grassmann thus clearly contributed to

[27] Shapiro himself, literally quoting Nagel (1939), acknowledges the contribution of Grassmann's theory of extension as a prefiguration of "the method of implicit definition" (Shapiro 1997, 147). On the relation between Peano's axiomatic vector theory and Grassmann's extension theory see in particular Dorier (1995, 247) and Cantù (2003, 331–338).

the development of abstract algebra, and, if universal algebra is conceived as a comparative investigation of different algebras—either to see what they have in common (Grätzer 1968, 7) or in connection with their interpretations in order to find a generalized notion of space that might serve as a uniform method of interpretation of the various algebras (Whitehead [1898], 1960, v, 29)—then he contributed to the development of universal algebra too. In particular, Grassmann's comparison of different structures was motivated by a foundational effort to distinguish different branches of mathematics according to the structural relations of their elements (§3.1.1). Yet one should remember that Grassmann did not understand algebraic systems as sets of given entities closed with respect to certain operations, and did not investigate classes of algebras, but only the different possible properties of operations.

If an essential condition for the development of methodological structuralism was the "transition from geometry as the study of physical or perceived space to geometry as the study of freestanding structures," a transition that was accomplished through the development of analytical geometry, projective geometry and non-Euclidean geometry (Shapiro 1997, 14), Grassmann's distinction between what we now call linear algebra (vector space theory in *n* dimensions) and geometry (the three-dimensional application of the former to physical space) (Grassmann 1845, 297) was certainly a relevant step, even if, as is often said, Grassmann did not bridge the gap between the discrete and the continuous, at least in the sense that he never considered real numbers as an extension of the system of rational numbers. On the contrary, he defined real numbers as being themselves magnitudes, thereby emphasizing the difference between discrete natural numbers and continuous real numbers, and grounding ET independently from arithmetic.

4.2. Methodological Structuralism

4.2.1. Mathematics Is Not the "Science of Quantity"

Methodological structuralism is often associated with a criticism of the definition of mathematics as science of quantity and number. Grassmann criticizes the traditional definition of mathematics as "science of quantity or magnitudes" (*Grössenlehre*) for two reasons. First, the word *Größe* refers only to continuous magnitudes and thus does not apply to the whole of mathematics.

> The name "theory of magnitude" is inappropriate for all mathematics, since one finds no use for magnitude in a substantial branch of it, namely combination

> theory, and even in arithmetic only in an incidental sense. (Grassmann 1995, 24)

That *Größe* refers only to continuous quantities is proved linguistically: in the German language *vermehren* and *vermindern* are connected to number, while *vergrössern* and *verkleinern* are connected to continuous quantities. Distinguishing what Wolff in *Mathematisches Lexikon* had not explicitly separated (Wolff translated both Latin terms *magnitudo* and *quantitas* by the same German word, *Grösse* [Cantù 2008]), Grassmann refuses to admit the reduction of geometry to algebra and to subsume continuous geometrical figures and real numbers under a single genus. Grassmann considers natural numbers as discrete quantities generated by repetition of a unit. Therefore the language rightly distinguishes numbers that increase or decrease from continuous magnitudes (including real numbers) that become bigger or smaller.

Second, the word *Größe* fails to express the main characteristic of mathematical objects, that is, that they are not given but generated according to a rule (§3.1.1). It is only in this second sense that Grassmann's remarks might be interpreted as having some relationship to structuralist approaches.

4.2.2. Mathematics Is Not about "Objects" but about Relations

A second fundamental feature of methodological structuralism is that mathematics is not about objects but about relations, or at least about objects only inasmuch as they are positions in a structure. Recalling what we have said about mathematics as the science of the particular, and especially about the role of the initial element in the "real" generation of mathematical abstract forms, it seems implausible to associate Grassmann with the conception of mathematics as the science of relations, notwithstanding Cassirer's and Hankel's tendency to do it. A further argument against this assimilation might come from some remarks by Banks, who insists on Grassmann's belonging to a German tradition that was interested in the development of a physical monadology in a Leibnizian sense (Banks 2013, 20–21), or the investigations by Brigaglia, who considers Grassmann to be the inspiring source for Segre's generalization of the notion of point (Brigaglia 1996, 159–160).

Yet there might be reasons to claim that, even if Grassmann's mathematics cannot be considered, *sensu stricto*, a science of relations, it might be an intermediate step between the traditional conception of mathematics and a structuralist approach. Such reasons are his constructivism and his consideration of operations as corresponding to pre-mathematical operations that can be applied to any kind of domain. Grassmann's constructivism is based on the idea that forms are the results of processes of connection, which construct or generate them, so that, in a dialectical perspective inherited from Schleiermacher, forms cannot really be distinguished from the process of their construction, and thus from the operations that occur in their concept formation and that determine their relations to other forms.

4.2.3. Mathematics Is the Study of "Relational Systems"

A third feature of methodological structuralism is the idea that mathematics investigates different "relational" systems, such as number systems, geometrical manifolds, various algebras, and so on. Again, Grassmann's separation between GTF, the specific branches of pure mathematics, and applied mathematics make it difficult to compare this approach to methodological structuralism. Certainly, he did not deeply investigate order relations, and he had a quite intuitive notion of continuous transformation. On the other hand, the effort to introduce a partition of mathematics that is based on different properties of the operations—an effort that became systematic especially in Robert's *Formenlehre*—or Grassmann's abstract analysis of different kinds of formal multiplications and their possible "realizations" in mathematical theories (Grassmann 1855, 216–217) can be considered a step toward the development of the project of a systematic investigation of relational systems.

4.2.4. Mathematics Is Not "Directly about the World"

There is at least one feature of methodological structuralism that Grassmann entirely subscribed to: it is the separation between pure and applied mathematics, which implies that mathematics is about abstract forms, and thus is not about the external world, or, in Grassmann's parlance, is not about a given that is not itself constructed by thought.

4.2.5. Mathematical Inferences Are "Formal"

A further feature of methodological structuralism is based on the idea that deductions are merely formal. Grassmann explicitly recognized that mathematical inferences are independent of intuition in the sense that primitive propositions can be conceived purely conceptually and that the only general logic law is the law of progression. Even if he admitted a relevant role of intuition as a heuristic tool, he never conceded that it should play a role in deductions. Yet Engel criticized Grassmann exactly because he did not manage to fulfill his project, maintaining an intuitive and unclear notion of continuous transformation (Grassmann 1844, 405). But this again is a controversial issue: Lawvere claims on the contrary that Grassmann's continuous transformation is unclear if wrongly conceived as a spatial translation, but that it becomes philosophically clear if it is understood as an action of the additive monoid of time.[28]

[28] "Grassmann philosophically motivated a notion of a 'simple law of change,' but his editors in the

4.2.6. Mathematics Goes toward Set Theory or Category Theory

If either set theory or category theory is a necessary condition for the development of methodological structuralism (an exception made, maybe, for Hellman's modal structuralism), then one should note that Grassmann did not contribute to the development of a theory of sets. On the contrary, his notion of product is incompatible with the modern understanding of a function, and his constructivism is incompatible with a set-theoretic perspective, where operations are defined on previously given sets of individuals (Cantù 2011, 2016). Grassmann did not contribute to the theory of category either, from a strict mathematical point of view, but Lawvere considers that category theory makes it "possible to recover some of Grassmann's insights and to express these in ways comprehensible to the modern geometer," and claims that Grassmann can be considered as precursor of category theory (Lawvere 1996, 255–256).

4.2.7. Mathematics Is Based on Some Kind of Axioms

We have seen that Grassmann did not present ET in an axiomatic way, at least not in a Hilbertian sense. But he considered mathematics to be based on concepts, because he deduced the differences between mathematical disciplines and mathematical forms by means of four fundamental concepts: equal, different, discrete, and continuous. And there is another sense in which GTF is based on some general conditions upon which a real operation might be called an addition or a multiplication (see §2.3.1). Finally, mathematics, being a formal science, does not have axioms in a traditional sense, but definitions. Yet the treatment of ET is not presented axiomatically, and GTF rather describes the operational features common to all operations that can be found in known mathematical disciplines, rather than an axiomatic description of algebraic structures.

4.2.8. Mathematics Studies Invariants

Structuralism is often associated to the investigation of invariant properties of different systems. GTF can be interpreted as a unifying perspective that studies what is invariant in different mathematical operations. Yet, as we have repeated many times, it is not a branch of mathematics. Some authors have tried to show that, even if Grassmann did not himself develop a comparison and classification of different mathematical systems by means of groups, his remarks on affine geometry influenced Klein's Erlangen program (see Engel 1911, 312, quoted in Toebies 1996, 120–122). Yet this is controversial, given that it has been argued

1890's found this notion incoherent and decided he must have meant mere translations. However, translations are insufficient for the foundational task of deciding when two formal products are geometrically equal axial vectors. But if 'law of change' is understood as an action of the additive monoid of time, 'simple' turns out to mean that the action is internal to the category A [of affine-linear spaces and maps] at hand" (Lawvere 1996, 255).

that Klein took his inspiration from Riemann and from projective geometry rather than from Grassmann and affine geometry (Rowe 2010, 142). Besides, Kannenberg has claimed that even if Grassmann individuated the group of circular and linear transformations starting from an analysis of different side conditions for the multiplication of extensive magnitudes, "the relation between groups and 'species of multiplication' is not reciprocal" (Grassmann 1995, 469).

To resume, Grassmann's GTF, which corresponds to the level of formal laws, is not part of mathematics, because it is underdetermined: it does not speak about a specific structure or a class of structures, but rather about the possible ingredients of a structure, being thus more similar to a metatheory of abstract structures. Grassmann's arithmetic contains just one kind of multiplication, whereas Grassmann's ET (the level of abstract content) concerns several kinds of multiplication, which can be fully determined only by further side conditions.

4.3. Philosophical Structuralism

In an earlier section of my chapter (see §3), I have tried to reconstruct Grassmann's conception from a perspective internal to his writings and to the spirit of his time. Now I will try to look at Grassmann's philosophy of mathematics from the present perspective, and thus look at some of the questions raised by Grassmann in the light of contemporary philosophical structuralism.

4.3.1. Grassmann's Claims on Structures

Even if Grassmann never used the term "structure" himself, I suggest that he might have agreed that (1) mathematical objects are characterized by structural properties; (2) structures are not given axiomatically; (3) general structures are distinguished from particular structures and from exemplars; (4) there is an interdependence between a structure and its objects, and (5) pre-mathematical operations between concepts are distinguished from operations in structures.

1. Mathematical objects are characterized by means of their relations and operations and by the relations between such operations (i.e., by structural properties). Structures are not themselves mathematical forms, because mathematical structures are universals, whereas mathematical forms, although being themselves concepts, are particulars.
2. Structures are not given axiomatically (construction versus postulation), and certainly not defined as in model theory by means of a domain and some relations and operations on it.
3. GTF is the study of relations and operations and concerns what we now call general structures (monoids, groups, rings). Mathematics is the study

of particular thought forms and concerns what we would now call particular structures. Applications of mathematics study particulars considered as given independently from thought and concern the investigation of exemplars of particular structures.

4. There is a dialectic between general structures, particular structures, and their exemplars, which allows further determination of mathematical forms as well as their relational properties. General structures do not exist independently from particular structures. Particular structures both determine and are determined by their objects. The distinction between GTF and mathematics concerns the question of the interdependence between relation and relata. In formal operations the operation might stand without its factors, whereas in mathematics it cannot. GTF is a sort of metatheoretical discourse on mathematical operations rather than itself a theory having mathematical structures as its objects.
5. There are some pre-mathematical relations and operations (equality, connection, and separation) that express some general operations of composition of concepts. They have some very general properties, such as substitutivity, commutativity and distributivity respectively. They are underdetermined with respect to mathematical operations, which have further properties: for example, the properties of the additive operation in an abelian group, or of the additive and multiplicative operations in a ring.

4.3.2. Grassmann's Claims Evaluated from the Perspective of Contemporary Philosophical Structuralism

If one evaluates the previous claims from the perspective of contemporary philosophical structuralism, one might remark that (1) there are no structures as universal objects in Grassmann; (2) there is no set-theoretic approach in Grassmann, (3) Isaacson's distinction between general and particular structures might apply to Grassmann's distinction between ET and arithmetic, (4) Grassmann's epistemology might be fruitfully compared to Parsons's non-eliminative structuralism as well as to (5) Feferman's conceptualism.

1. No universals: Grassmann's epistemology suggests the need for a constructivist alternative to the *ante rem* / *in re* ontology, an alternative that might speak about structures without considering them as mathematical objects, and especially not as universals.

Grassmann certainly had an ontological perspective, at least in the sense that he was an idealist and a constructivist: mathematical forms have an objective nature. All products of thought processes become objective in the moment of their construction, and can thus be successively taken as given (Grassmann 1844, 22). For this objective nature of thought forms, it is certainly not easy to associate Grassmann with an eliminative (nominalistic) position à la Hellmann (1990). Nor does it seem possible to consider thought forms as *in re* universals (as in Shapiro's account of eliminative structuralism [1997, 9]).

Yet the question about Grassmann's structuralism can be asked once more at another level, that is, at the level of GTF. Grassmann could be associated with an eliminative approach at this level, because there are no such things as the objects of a general structure. General structures are not the genus of which spaces, number systems, and so forth are species (Burgess 2015, 107–108), but are based on underdetermined concepts that get their full determination once applied to particulars, and it is this very application that makes further side conditions explicit and allows for the determinateness that is needed to treat something as an object of mathematics.

2. No set-theoretic notion of structure: a preliminary objection might concern the anachronism of applying a philosophical perspective that is grounded on different notions of function, object, and concept. According to Grassmann, operations are not closed on a domain, either because the domain might be considered variable (in mathematics) or because the operations might be considered independently from their factors or from a domain on which the factors should vary.

The main problem in the case of Grassmann is to exactly determine what he might mean by "structure." Whereas the model-theoretic notion of structure is based on a domain (a set) to which the operation is applied (and the definition of the structure concerns this domain, at least inasmuch as it has properties of closure with respect to operations), there is not even the possibility of determining closure properties in Grassmann's consideration of formal operations.

3. General and particular structures (Isaacson and Shapiro): a comparison with Isaacson's structuralism is interesting in order to appreciate another aspect of Grassmann's structuralism: the distinction between formal and real operations. Isaacson's structuralism is antithetic to Grassmann's perspective, at least inasmuch as it defends the existence of structures but not of mathematical objects (structures themselves are not mathematical objects), and he centers his perspective on axiomatic postulation rather than on construction.

Isaacson distinguishes between general and particular structures. The distinction is derived from the way we linguistically refer to them, either by the determinate article (the structure of natural numbers) or by the indeterminate article (a group) (Isaacson 2011, 2–3, 18). Isaacson remarks that Bourbaki believed that the mathematical interest was mainly on the side of general structures, and Grassmann might agree on that point.

Yet, according to Isaacson, the philosophical interest is all on the side of particular structures, because structuralist realism concerns the existence of particular structures. This might be related, I think, to Grassmann's choice to consider GTF as not properly belonging to mathematics: it concerns general and not particular structures. Besides, Isaacson notes that particular structures can themselves be classified into abstract and concrete structures (type and tokens), being in the relation one-many. This might correspond to Grassmann's distinction between vector space theory and 3-dimensional geometry, or between the abstract real level and the level of applications.

Shapiro had introduced a distinction between "algebraic" and "non-algebraic" fields of mathematics, that is, between mathematical subdisciplines that concern a class of structures or a single structure respectively.[29] Even if one might claim that algebraic fields are about general structures, whereas non-algebraic fields are about particular structures, the use of Shapiro's distinction is problematic, because it does not do justice to Grassmann's idea that all mathematical fields are about particular structures. It is only GTF that concerns general structures. This is an important aspect of what we might call Grassmann's *concept structuralism*, as opposed to an *object structuralism*, which requires a complete determinateness of the objects and therefore an identity criterion.[30] And this might explain why Grassmann would probably disagree with the idea (shared, e.g., by Isaacson

[29] See Shapiro (1997, 40–41). The distinction made by Grassmann between arithmetic and ET can be compared with Shapiro's distinction between non-algebraic (e.g., arithmetic and analysis) and algebraic fields (e.g., group theory, field theory, or topology, which are about a class of related structures).

[30] More should be said on this notion of "concept structuralism," but this would require a new article. For the sake of the understanding of Grassmann's perspective, it might suffice to say what concept structuralism is not, and how it is related to a dynamic process of mathematical determination of pre-theoretical notions. (1) Concept structuralism is not a historical tradition like "conceptual mathematics" (see, e.g., Stein 1988 and Ferreirós 2007). (2) Concept structuralism is not necessarily characterized as a weak form of Platonism (see, e.g., Ferreirós's effort to define conceptual structuralism as a version of weak Platonism, suggesting that structures exist as abstract entities but are not necessarily independent from the mathematician). Structures are conceptual tools that describe general properties of the operations among particular entities. In a proper sense, only the particulars can be said to exist as fully determined objects. (3) Concept structuralism is based on the idea that mathematics is a dynamic process that tries to further determine some pre-theoretic notions, e.g., by considering the algebraic closure of an underdetermined operation, so that mathematical objectivity is ultimately grounded in processes of concept formation. I would like to thank José Ferreirós for the rich discussion we had on the topic, and for the useful insights I got from the reading of his essay on mathematical practices (Ferreirós 2016), and his unpublished manuscript on Feferman's conceptualism (Ferreirós 2018).

and Shapiro) that the philosophical problem consists in accounting only for the existence of particular structures.

Another interesting point is Isaacson's remark that there cannot be objects without particularity and without an identity criterion: therefore Shapiro has a problem when he pretends to speak about the objects of a structure (as is proved by Keranen's objections). I take Isaacson's remark to suggest that whenever structures are introduced axiomatically (or by postulation), then one cannot talk about mathematical objects of these structures, because no identity criteria are available. Grassmann avoids introducing vector space systems by postulation, exactly because he believes that they concern mathematical forms whose construction is determined by their generating laws, which also allow for identity criteria. Construction rather than postulation has for Grassmann a foundational value. This position is again antithetic to the position of Isaacson, who believes that only postulation has foundational value, and that construction was fundamental only in the logicist perspective, because the construction should prove the logical nature of mathematical concepts.

4. Parsons's non-eliminative structuralism: Grassmann's approach can be interestingly compared with Parsons's version of non-eliminative structuralism.[31] Mathematical objects are taken to be particular forms (e.g., numbers, extensive magnitudes, etc.). Neither formal operations nor structures themselves seem to be considered mathematical objects, because they appear in GTF as underdetermined, devoid of an identity criterion, which on the contrary seems to be a necessary condition for something to be an object (Isaacson 2011). Talk about formal operations is rather metatheoretic, and general structures (in Grassmann's sense) are not even deficient-property objects (Burgess 2015), because they are not structures whose elements have no individual nature, but rather a bunch of operations considered independently from their "application" to particulars. There is a dialectic between particular structures and their exemplars, as in the case of the geometric analogy that guides the development of Grassmann's ET. Similarly, Parsons considers structures to be not

[31] A general classification of all kinds of contemporary variants of structuralism is not available, and various terminologies conflict one with the other. I will adopt Parsons's terminology, and distinguish *eliminative* from *non-eliminative* structuralism: "Eliminative structuralism . . . proposes some procedure for paraphrasing the language that refers to the objects we are concerned with, usually either the numbers of one of the number systems, or sets, so that commitment to the objects concerned, even the conception of them as a distinctive kind of object, disappears. . . . [Non-eliminative structuralism] takes the ideas behind structuralism not as the basis for a program for eliminating numbers, sets and other pure mathematical objects, but rather as the basis for an account of them as objects, as the objects which theories of numbers and sets talk about when taken more or less naïvely" (Parsons 2004, 57).

free-standing but connected to instantiations developed in mathematical practice. Grassmann's vector space theory is presented in a purely abstract way in the first edition of the *Ausdehnungslehre*, but a geometric analogy guides the development of ET. This dialectic between the particular structure and one of its exemplars suggests a comparison with Parsons's claim that talking about mathematical objects is legitimate in structuralism, even if their identity criteria cannot be established exclusively by means of structural properties, but require some reference to extra-structural properties.

Grassmann similarly believes that it is possible to talk both about operations that are only partially determined and about operations that are fully determined in some particular structure or in an exemplar of it. This is legitimate, because, according to Parsons, structures are not free-standing but are somehow connected to instantiations developed in mathematical practice.

With Parsons, Grassmann would agree that mathematical objects such as natural numbers are usually given in a realization of the structure, and that "some mutual dependence in understanding what the objects of a domain are and what their most important properties and relations are" need not be circular (Parsons 2004, 73). I suggest that Grassmann would understand in a dialectical way the relation between the so-called intended model and the axiomatic formulation of a structure.

Grassmann's perspective cannot be compared with Parsons's Quinean approach, according to which "speaking of objects just is using the linguistic devices of singular terms, predication, identity, and quantification to make serious statements" (Parsons 1982, 497). Yet I think Grassmann shares what I take to be a presupposition of Parsons's structuralism: the possibility of talking both about objects that are only partially determined (e.g., determined only by their structural properties, even when this does not allow us to distinguish objects in the structure, as might be the case for i and $-i$ in the structure of the complex numbers) and about objects that are fully determined in some instantiation of the structure (where one might have identity criteria or knowledge of specific relations between the objects).

5. Feferman's conceptualism: Feferman's conceptual structuralism is based on the belief that the general ideas of order, succession, collection, relation, rule, and operation are pre-mathematical. Likewise, Grassmann's conceptual constructivism distinguishes pre-mathematical operations between concepts (some general notions of composition) from mathematical operations.

According to Feferman, the basic objects of mathematical thought exist only as abstract mental conceptions resulting from processes that are independent of the concrete objects to which they are applied, and based on pre-mathematical concepts such as relations, rules. and operations (Feferman 2009). Grassmann might substantially agree on several of Feferman's 10 theses that characterize his version of conceptual structuralism.[32]

As in Feferman's version of conceptual structuralism, mathematics does not concern only universal or relational concepts, but also particular concepts considered as autonomous thought forms. The focus is on the procedures of concept formation.

4.3.3. Grassmann's Challenges to Contemporary Structuralism

The comparison between Grassmann's epistemology and contemporary philosophical structuralism can be used both to better understand Grassmann's philosophy and to consider whether new challenges might derive from his "obsolete" perspective.

Grassmann certainly contributed to the development of methodological structuralism. He criticized the traditional definition of mathematics as a science of magnitudes, and even if he still associated it with particular thought forms, he considered the latter to be determined by their generating law applied to an initial element. Grassmann clearly separated pure from applied mathematics, and developed a formal analysis of certain properties of connections that can be found in all mathematical branches. Even if, *sensu stricto*, he did not axiomatize mathematics, he individuated certain side conditions of the general connections that can be considered as invariant under specific kinds of transformations.

From a philosophical perspective, Grassmann's general theory of forms and the general definition of multiplication that occurs in ET can be interestingly compared to a non-eliminative structuralism associated with a constructivist ontology, as for example Parsons's or Feferman's structuralism. With the latter Grassmann would share the idea that the basic objects of mathematical thought exist only as abstract mental conceptions resulting from processes that are independent of the concrete objects to which they are applied, and based on pre-mathematical concepts such as relations, rules, and operations (Feferman 2009). With the former Grassmann would share the idea that mathematical forms (including numbers and extensive magnitudes) are the objects that mathematics talks about (Parsons 2004, 57).

Even if most questions related to the development of philosophical structuralism, such as Benacerraf's dilemma on natural numbers, cannot really be

[32] See in particular Feferman's theses 1, 2, 3, 8, and 9 (2009, 3).

compared with Grassmann's pre-set-theoretic approach, the epistemological challenge is taken into account in his constructivism. So Grassmann's most interesting contributions to contemporary structuralism might be seen in several challenges: (*a*) find a constructivist alternative to the *ante rem* / *in re* ontology, (*b*) verify whether a form of conceptualism might explain how mathematicians talk about structures without wholly abstracting from their instantiations, (*c*) consider the foundational role of series in mathematical and scientific thought, (*d*) develop an investigation of the differences between what we have called concept structuralism and object structuralism.

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