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Pasch's Empiricism as Methodological Structuralism

Dirk Schlimm

1. Introduction

A fundamental tension in philosophy of mathematics, one that goes back at least to Plato's *Meno*, is that between a view of mathematical entities as being abstract in nature and a view of knowledge as being of concrete (or causal) origin. Considered independently, each view can be quite appealing, but their combination raises the serious difficulty of giving a coherent account of mathematical knowledge. Abandoning, or at least substantially weakening, one of these views is a common move to resolve this dilemma. One thinker who resisted the urge to give up his conviction of the empirical origins of human knowledge was Moritz Pasch (1848–1930).

Throughout his life Pasch referred to his own philosophical outlook as *empiricist*. When he proclaimed in the introduction to his *Vorlesungen über neuere Geometrie* that "geometry is seen as nothing else but a part of natural science" (Pasch 1882b, 3), he meant this to be understood literally, in the sense that mathematical theories are based on empirical concepts. Geometric points, for example, are introduced at the beginning of his book as those physical objects that cannot be divided any further within the limits of what we can observe. Likewise, Pasch rejects the common demand that geometric lines should be imagined as being infinitely extended, since this precludes them from being (at least in principle) perceptible objects. Instead, he considers finitely extended line segments to be among the primitive objects of geometry (Pasch 1882b, 4). For him, all mathematical propositions, i.e., not only those of geometry, are ultimately formulated on the basis of observations of physical objects, and he maintains that we can understand the basic mathematical terms only by indicating appropriate objects ("den Hinweis auf geeignete Naturobjecte," Pasch 1882b, 16).[1] The further development of mathematics then proceeds through the deduction of propositions

[1] In this context, one also speaks of "ostensive definitions" of the primitive terms.

Dirk Schlimm, *Pasch's Empiricism as Methodological Structuralism* In: *The Prehistory of Mathematical Structuralism*. Edited by: Erich H. Reck and Georg Schiemer, Oxford University Press (2020). © Oxford University Press.
DOI:10.1093/oso/9780190641221.003.0004

and the definition of new concepts, and both of these processes sustain the epistemological status of their starting points.

Pasch's empiricist view of mathematics appears to be at odds with one of the basic tenets of structuralism, according to which mathematics is about purely abstract structures. Nevertheless, I will argue in the remainder of this chapter that Pasch's mathematical work drove him to adopt an approach that can justly be called "structuralist," in spite of the fact that his deeply held philosophical convictions seem to be incompatible with it. In order to do this, I will begin with discussing the notion of "methodological structuralism" (Reck 2003) and propose two minimal conditions that an approach has to satisfy to qualify as being structuralist (section 2). I will then look in more detail at Pasch's work in geometry (section 3) and the foundations of arithmetic (section 4) to ascertain that it does indeed satisfy the proposed conditions for minimal methodological structuralism. Thus, I conclude that Pasch's approach has its rightful place in an account of the prehistory of mathematical structuralism.

2. Minimal Methodological Structuralism

The notion of "methodological structuralism" was introduced by Reck (2003, 371) in order to distinguish the more ontologically oriented views on the nature of mathematics, like those expressed by Resnik (1997) and Shapiro (1997), from a certain way of practicing mathematics that is (in principle) independent of one's particular ontological commitments.[2] To assess whether a structuralist methodology can be found in the investigations of Pasch and others, it will be useful to identify some of its characteristic features.

A paradigmatic example of a structuralist methodology is the work in modern abstract algebra as presented by van der Waerden (1930). Reck describes this as follows:

> What modern algebraists do is to study various *systems of objects*, of both mathematical and physical natures (the latter at least indirectly), which satisfy certain general conditions: the defining axioms for groups, rings, modules, fields, etc. More precisely, they study such systems *as* satisfying these conditions, i.e., as groups, rings, etc. (2003, 371)

Thus, while an algebraist might explicitly discuss the field of complex numbers in her work, only those properties that are formulated in the field axioms and those

[2] See also the editorial introduction to this volume by Reck and Schiemer.

that follow from them are considered. That only the relations that are specified by the general conditions that define these systems are taken into account, but not any other properties that these objects might have as individuals, is what makes this approach *structural*. Here is how Reck formulates this idea:

> a methodological structuralist will not be concerned about the further identity or nature of the objects in the various systems studied. He or she will simply say: Wherever they come from, whatever their identities and natures, in particular whatever further "non-structural" properties these objects may have, insofar as a system containing them satisfies the axioms . . . , the following is true of it: . . . This is the sense in which methodological structuralism involves a kind of *abstraction*. Here abstraction concerns simply the question of which aspects of a given system are studied and which are ignored when working along such lines. (Reck 2003, 371)

Notice that for a methodological structuralist "abstraction" is not necessarily understood as a process that yields some kind of new abstract entities, but rather as an attitude of restricting oneself to taking into account only some features of the systems under investigation, while disregarding others. In sum, methodological structuralism can be described as the study of systems of objects that are characterized, or defined, axiomatically, with an exclusive focus on the relations that hold between these objects, while ignoring further questions about the nature of the objects. Dedekind's *Was sind und was sollen die Zahlen?* (1888) is a perfect example of an approach that falls under this definition (see Ferreirós and Reck in this volume). However, the insistence on axiomatic definitions seems to be too strong, and Reck himself adds the qualification that methodological structuralism is only "*typically* tied to presenting mathematics in a formal axiomatic way" (Reck 2003, 371; my emphasis). We should also note that the second condition formulated previously (i.e., the *focus* on relations) leaves open the possibility of pursuing structuralist investigations at one time and working along other, nonstructuralist lines at other times. Thus, methodological structuralism can be one particular approach among others pursued by the same mathematician; an approach that can be taken in certain investigations, but that can be ignored in others. It is the result of an attitude about how to conduct certain investigations that can be independent of one's philosophical conceptions of mathematics and the nature of mathematical objects.

With the refined understanding of methodological structuralism given in the previous paragraph we must confront the problem of triviality: Is anything at all excluded by the characterization or has now every mathematician become a methodological structuralist? For example, Euclid can be interpreted as having studied a system of points and lines in his planar geometry, taking into consideration only those relations between them that were licensed by his axioms. This suggests a further criterion to distinguish an approach that is explicitly intended to be structural from one in which axioms are used to describe a single system that is being studied.[3] On the one hand, Euclid investigated only one particular system, which consisted of idealized points and lines, and it seems fair to say that he did not envisage other systems of objects to satisfy the same relational properties. For Dedekind, on the other hand, it was clear that the natural numbers were only one particular instance of a simply infinite system and that there were others as well, like the system of his potential thoughts (*Gedankenwelt*). Similarly, in modern algebra groups and fields can be instantiated by many different systems, like numbers, rotations, etc. Based on these reflections, I propose the following two conditions that must be satisfied by investigations to count as being along the lines of a minimal version of methodological structuralism:

(1) Focus on *relational features* of systems of objects.
(2) The possibility of *multiple* systems that share these relational features must be envisaged.

With these two conditions in hand, we can now look at the works of particular authors and assess whether they qualify as being structuralist in methodology.[4]

3. Empiricist Structuralism in Geometry

Various aspects of Pasch's work in geometry appear to be congenial to methodological structuralism. Pasch presented in his *Vorlesungen über neuere Geometrie* (1882b) the first axiomatization of projective geometry in a way that is considered to be rigorous by contemporary standards. Indeed, Hilbert's axiomatization of Euclidean geometry, *Grundlagen der Geometrie* (1899), can be readily interpreted along structuralist lines (see Sieg's article on Hilbert in this volume) and was heavily influenced by Pasch. Moreover, Pasch famously also gave a characterization of the nature of deduction that emphasizes the relational features of the systems under investigation and which is worth quoting in full:

> In fact, if geometry is genuinely deductive, the process of deducing must be in all respects independent of the *sense* of the geometrical concepts, just as it must

[3] For a discussion of various roles and functions of axioms, see Schlimm (2013a), in particular 49–52 for their descriptive function.

[4] As far as I can tell, the approaches of the authors presented in this volume all satisfy the conditions for minimal methodological structuralism.

> be independent of figures; only the *relations* set out between the geometrical concepts used in the propositions (respectively definitions) concerned ought to be taken into account. (Pasch 1882b, 98)[5]

Pasch's insistence that, in order to be rigorous, deductions must be independent of the meanings of terms and instead rely only on their relational connections, as opposed to the particular meanings of the concepts, forms the cornerstone of his *deductivism*, which he himself referred to as "formalism" (Pasch 1914, 121). This approach meets the first condition for minimal methodological structuralism and thus appears to point to a general structuralist understanding of mathematics. In fact, his axiomatic standpoint has been interpreted as foreshadowing the idea that a system of axioms implicitly defines an abstract structure (Tamari 2007, 6 and 96). According to these indications, it seems straightforward to consider Pasch a methodological structuralist. However, Pasch also held an *empiricist* philosophy of mathematics, a brief sketch of which was given in the introduction to this chapter, which stands in stark contrast to the interpretation of axioms as implicit definitions and requires us to take a closer look at his works and adopt a more nuanced position.

The empiricist standpoint, according to which the fundamental concepts and propositions of mathematics are empirical in nature, is the background for most of Pasch's works, not only in geometry, but also in analysis. For Pasch, the basic, or "core" (*Kern*), propositions that form the starting points of a deductive presentation of mathematics are "directly based on observations" (1882b, 17) and "obtained through experience" (1914, 3). The content of a mathematical discipline like analysis, Pasch maintains, is constituted by facts; these can be derived from basic facts, which are themselves expressed by the basic propositions (1914, 3). However, despite his insistence on the empirical foundation of mathematics, Pasch quickly realized that a deductive development of mathematics cannot be carried out on the basis of empirical facts alone. This led him to distinguish between a mathematical set of axioms called a "stem" (*Stamm*) and a philosophically grounded, empirical set of axioms (first called "basic principles" and later a "core").[6] One of the reasons for this distinction was the observation that the axiomatic presentation of a mathematical theory does not necessarily determine the meanings of its primitive terms in a unique way. This insight was not based on some considerations of first-order logic or nonstandard models as we might be inclined to think nowadays, but on the duality of projective geometry, which was identified in the 1820s by Poncelet and Gergonne (Pasch 1914, 142). Duality

[5] All translations are by the author; translations of Pasch (1920b) and Pasch (1921) are based on those of Pollard (2010).

[6] See Schlimm (2010).

is the curious mathematical phenomenon in which, if the primitive terms (say of "point" and "line," and the relations "lying on" and "contains") of a theorem of projective geometry are interchanged, the result is again a theorem of projective geometry. In Pasch's words, the stem propositions for this discipline form a collection of propositions that is "transformed into itself" if the stem concepts of point and line are interchanged.[7]

> This fact, which is the source of duality, provides the proof that the group of projective stem propositions may not be considered as a definition of the projective stem concepts. Rather, it shows how the relations that are expressed by the projective stem propositions can be satisfied in more than one way. (Pasch 1914, 143)

Thus, the form of the axioms does not determine whether the term "point" indeed refers to points or to lines and, because the axioms of projective geometry cannot fix the meanings of the terms themselves, they cannot be regarded as their definitions.[8]

While some concepts may be defined by the propositions in which they occur, Pasch observes that it is not possible that all concepts could be defined in this way, because this would allow the possibility "that definitions can generate mathematical concepts out of nothing" (1914, 143). He elaborates:

> If one would want to claim that a totality of relations σ between concepts β, e.g., the basic propositions of arithmetic, could constitute a definition of the totality of concepts β, then one would have to be certain that the relations σ could not be satisfied in any other way than by the concepts β, excluding also the case where the concepts β are permuted. (1914, 143)

What Pasch explicitly rejects here is the understanding of a set of axioms (which govern the relations σ) as defining the primitives occurring in them (which refer to the concepts β), which is commonly referred to as an implicit definition. In fact, in reference to the first edition of Schlick's *Allgemeine Erkenntnislehre* (1918), which discusses Hilbert's approach to definitions by axioms, Pasch writes that

> the expression "implicit definition" has a different meaning when used by Mr. Schlick (definition by axioms). I have presented in §72 of *Veränderliche und*

[7] See Eder and Schiemer (2018).

[8] A similar argument is made by Frege in his correspondence with Hilbert (Frege 1976, 58–80).

> *Funktion* [i.e., Pasch 1914, 142–143] the concerns that speak against a definition by axioms. (1920a, 145)

Readers should note that Pasch himself uses the term "implicit definition" in his writings, but in a different sense, namely in the sense of contextual definition, not in relation to axioms.[9] In particular, Pasch does not allow implicit definitions for the basic terms, but only for the introduction of new terms using basic or already defined terms. An implicit definition, in Pasch's sense, tells us how to replace an expression that contains a new term by an expression that does not contain it.[10] Pasch contrasts them with explicit definitions, whereby something that belongs to a genus is defined by specific marks (Pasch 1914, 20). Thus, Pasch's understanding of definitions, which is rooted in his empiricism, is clearly at odds with interpreting him as understanding axioms as implicit definitions of the class of their models or of an abstract structure.

In light of the preceding considerations, we can see how Pasch's move of distinguishing basic concepts and propositions from stem concepts and propositions allowed him to keep a deductivist view of mathematics (according to which a mathematical discipline is developed deductively from its stem), while at the same time retaining his convictions about empiricist foundations for mathematics (for the core). For Pasch, to demonstrate the viability of his empiricist philosophy of mathematics in general, each stem had to be connected to a core. For the case of projective geometry, Pasch showed in his *Vorlesungen* how the stem concepts and propositions can be linked to their empirical basis; I have referred to this project as "Pasch's Programme" (Schlimm 2010). In addition, because in a purely deductive development of a theory the stem propositions of a discipline can play the role of basic propositions (Pasch 1914, 121), Pasch can accommodate the observation that mathematicians can disagree on their philosophical views on the nature of mathematics while at the same time agreeing on the validity of proofs and theorems. After all, from a purely logical point of view, a theory can be developed from either a core or a stem, as long as they are consistent (Pasch 1924, 232).

One reason for Pasch's insistence on an empirical foundation of mathematics is his concern for its use in scientific and everyday applications.

> To apply mathematics, the basic concepts must refer to something that is present in the world of experience and for which the content of the basic propositions is meaningful and valid. We acknowledge this connection with experience as soon as we consider analysis to be something else than . . . an

[9] See Gabriel (1978) and Pollard (2010, 36–39).

[10] His introduction of the term *Menge* (set) is an example (Pasch 1914, 19).

> internally consistent construction [*einen Bau von innerer Folgerichtigkeit*]. (Pasch 1914, 138)

Thus, although Pasch allows for the possibility of working with meaningless terms in mathematics (if the stem is left uninterpreted), or with terms that refer to something other than empirical concepts and relations, he believes that a complete picture of mathematics should include an account of its applicability and that this is best given by empiricism. In addition, the latter removes any doubt about the arbitrariness of mathematics.

> The traditional view renders the mathematical point as a concept that does not refer to something real; I would like to call it a *hypothetical concept*. . . . Now, if hypothetical concepts and the assumed relations between them (hypothetical propositions, hypotheses) are applied to objects of nature, at first to drawn figures, then this remains something arbitrary as long as we do not formulate the laws that govern this application; hereby one has to put up with the imprecision that inheres in the application. It then becomes necessary to make *two different kinds of hypotheses*. Hypotheses of the first kind, which are those already mentioned, only relate the hypothetical concepts with each other, not with empirical ones; hypotheses of the second kind are to establish a bridge between hypothetical and empirical concepts. Compared to the empiricist way of proceeding this is nothing but a detour. (Pasch 1914, 139)

In short, the need for additional hypotheses that connect the mathematical stem concepts to their empirical counterparts when mathematics is applied is used as an argument in favor of using empirical concepts from the start. Notice how Pasch anticipates the need for connecting the scientific terms of a hypothetico-deductive theory with empirical referents; without any reference to Pasch, later philosophers of science referred to his hypotheses of the second kind as "coordinating definitions" (Reichenbach 1928, 31), "bridge laws" (Nagel 1961, chap. 11, sec. 2.3), or "bridge principles" (Hempel 1966, 72). Considerations of parsimony lead Pasch to skip the bridge laws and use empirical hypotheses directly.

Now, where does this discussion leave Pasch with regard to structuralism? He certainly would disagree that mathematics is about abstract structures. However, he would allow us to hold this view if we wanted to, but at the cost of having to explain how these structures can be applied to the world. Pasch himself clearly prefers an empiricist account of mathematics for which the problem of application does not arise. Nevertheless, his mathematical practice satisfies the two conditions for methodological structuralism laid out at the end of section 2: The focus on the relations that are expressed by the axioms of a mathematical discipline, not on the nature of its elements, is what guarantees the rigor of mathematical deductions. Mathematicians can develop their theories on the basis of stem concepts and propositions, which need not have a determinate reference, but can have multiple realizations instead; the paradigmatic example of such a theory is projective geometry. In fact, it was the duality of projective geometry that led Pasch to the distinction between a philosophically meaningful axiomatic foundation (consisting of core concepts and core propositions) and a mathematically sufficient axiomatic basis (consisting of stem concepts and stem propositions).

4. Empiricist Structuralism in Arithmetic

We have seen in the previous section how Pasch's insistence on rigorous deductions together with the surprising fact of the duality of projective geometry pushed him toward a minimal version of methodological structuralism, which he was able to combine with his empiricism about mathematics by separating purely mathematical axioms from philosophically grounded ones. In the present section I want to look at Pasch's work in a different mathematical discipline, namely arithmetic, in order to illustrate that the previous considerations were not unique to geometry, but arose also in other disciplines. This suggests that one ingredient for the emergence of structuralist views of mathematics was a particular attitude towards rigorous deduction that was developed in the 19th century.[11]

In addition to geometry, Pasch also worked on the foundations of arithmetic throughout his entire career; e.g., see Pasch (1882a, 1909, 1914, 1921, and 1924). The development of mathematical concepts on the basis of empirical ones through definitions and deductions, which Pasch presented for geometry, is the same approach he adopted for establishing the foundations of number theory and analysis. Here, too, Pasch aims at reducing the discipline to a core from which everything else can be derived. For him, such a reduction serves to present mathematics as a deductive discipline, justifies confidence in its consistency, allows us to assess its certainty, and forms the basis for any philosophical reflection about mathematics, such as the question of its relation to experience (Pasch 1921, 155).

Pasch disagrees with "the standard practice of putting a more or less finished notion of number at the beginning" of one's mathematical investigations (Pasch 1921, 155).[12] Instead, in his account "the natural numbers do not appear all of a

[11] See Gray (1992) and Detlefsen (1996) for a general overview of these developments.

[12] Contrast this with, for instance, the famous saying attributed to Kronecker that "God created the whole numbers, everything else is the work of man" (Weber 1893, 15).

sudden: they stand at the end of a long and difficult path" (Pasch 1920b, 4). Let us now briefly examine Pasch's account of natural numbers, as presented in his work on the origin of the concept of number.[13] Pasch's empiricist outlook is formulated clearly in the very first paragraph:

> The sort of thought process to be exhibited here might arise in any person who, first, considers only the *things* he himself perceives and distinguishes one from another and who, second, credits himself with eternal life and unlimited memory. Among the things observed by this person are his own actions. (Pasch 1920b, 1)[14]

We notice immediately that Pasch goes beyond assuming what is humanly possible, but instead posits an ideal agent with human-like cognitions, perceptions, and actions, but endowed with "eternal life and unlimited memory." While this move might seem striking at first, it has been popular among empiricists, who would otherwise have to restrict themselves to a finite (and in fact rather small) number of experiences; for example, a very similar starting point of a contemporary empiricist account of mathematical knowledge is Kitcher's *ideal subject* (Kitcher 1983, 109–111). As a careful systematizer, Pasch singles out 11 core concepts to describe the actions of the ideal agent: (1) *things*, (2) *proper names*, and (3) *collective names*, which are themselves things; the actions of (4) *specifying a thing*, (5) *assigning a proper name*, and (6) *assigning a collective name*—collective names can only be assigned to collections of things that were previously specified or assigned a proper name by the agent; any such action is (7) an *event*, which can be temporally related to other events by the relations (8) *earlier*, (9) *later*, and (10) *immediate successor*; finally, an ordered sequence of events forms (11) a *chain of events*. By considering names and events (both of which he considers to be things) in addition to physical objects Pasch frees the ideal agent from being restricted to what is physically present, and by considering experienced events the ideal agent is able to introduce order:

> I assume that I have experienced some events on which I confer the collective name *A*. By experiencing these events, I have registered observations about succession and immediate succession, about precedence and immediate precedence. But the events *A* also produce in me a comprehensive concept that

[13] Pasch's "Der Ursprung des Zahlbegriffs" was completed in 1916 and appeared in print in two parts, Pasch (1920b) and (1921), which were reprinted together in Pasch (1930a). The approach is based on the account given in Pasch (1909), but contains several modifications.

[14] The English translations in this section are taken from Pollard (2010).

> combines them into a whole, into a thing that I call *the chain of the events A* or, more briefly, $\mathfrak{A}$. (Pasch 1920b, 17)

So, while we may think of the events A as something like a finite set, $\{s_1, s_2, \ldots, s_n\}$, the chain $\mathfrak{A}$ of events A is more like a finite ordered set: $< s_1, s_2, ..., s_n >$. Given that Pasch allows the same thing to be given different names and be specified multiple times, he introduces the notion of a *line* for those chains whose elements are all specifications of different things.[15] The *members* of a line are those things that are specified by the elements of the line. Using these notions, Pasch introduces the concept of *number* as follows: to determine the number of a given collection N, first an arbitrary larger line $\mathfrak{Z}$ is obtained, whose members have the collective name z and whose first member is called e. Then,

> from among the members z that follow e I can specify one and only one member n such that the segment of $\mathfrak{Z}$ reaching as far as n is equivalent to the collection N.
>
> In addition to N, all and only the collections that are equivalent to N yield this member of the line $\mathfrak{Z}$.
>
> The thing n is called the *number* drawn from the line $\mathfrak{Z}$ for the collection N.
>
> Any z other than e can serve as "numbers." (Pasch 1921, 149)

After extending the use of the term "number" also to the member e of $\mathfrak{Z}$, and introducing the names "one," "two," "three," etc., for the members of $\mathfrak{Z}$, Pasch concludes:

> Now all the members of the line $\mathfrak{Z}$ have become numbers. Notches in a stick can serve as members of such a line. One notch must be singled out as the first, with all the remaining notches appearing to one side of it. The next member of the line is always the next notch over. (Pasch 1921, 150)

On the one hand, Pasch's example of notches on a stick nicely illustrates the empirical character that the natural numbers have for him; on the other hand, it also illustrates that for him the numbers are not one single, particular system of objects. In fact, it is compatible with this account that Julius Caesar is one of

[15] In (1909) Pasch used the terms *Folge* and *Reihe* (sequence and series), but he changed them in (1920b) to *Kette* and *Rotte* to avoid imbuing terms that already have multiple mathematical meanings with new meanings (Pasch 1920b, 17 and 19). While *Kette* translates straightforwardly as "chain," the term *Rotte* is less familiar and thus more difficult to translate. In a military formation, a *Rotte* consists of those soldiers or planes that are side by side; in this case an individual is called a *Glied*. Accordingly, Pollard (2010, 68) translates *Rotte* as "line" (and *Glied* as "member"), which we follow here, despite the fact that Pasch wanted to use a term that does not already have a mathematical meaning.

the members of 𝔷, and thus a number.[16] If a collection is empty, then there is no member of 𝔷 that can serve as the number of this collection. For this case, Pasch introduces the name "zero" as if it were the name of a thing (using an implicit definition, in Pasch's sense).

Pasch continues his account by introducing the figures "0", "1", . . . , "9", together with rules for obtaining greater numerals (technically, these are chains of specifications of figures) as distinct names. In this way only simple combinatorial processes are required to generate a potentially infinite list of names for numbers.

> For each number drawn from 𝔷, the figures yield a sign [*Zeichen*], and the sign yields a name. So figure-chains will satisfy our need for numerical signs in every case. . . . Conversely, any figure-chain one cares to construct can serve as a numerical sign, as long as I pick a sufficiently "large" 𝔷. (Pasch 1921, 152)

Thus, the system of numerals is a systematically obtained sequence of names that can be used to refer to the members of any chain of things that one decides to use as numbers. In the first exposition of this way of proceeding, Pasch leaves it at that, switching effortlessly and without much ado from numbers as things to their names (e.g., "If a number, (i.e., its name) consists only of nines, . . ." (Pasch 1909, 35); he understands a calculation to be the determination of a fixed name (e.g., in the decimal system) of a number that is given by an arithmetical expression (Pasch 1909, 53). Five years later, in 1914, Pasch is more careful and gives more explicit explanations. After noting that the construction of decimal place-value numerals yields names for each desired number, he notes:

> Once this is achieved, one can disregard which things and which chain of these things were originally used; one only needs to hold fixed *the names of these things, of the numbers*. . . . The decimal place-value name of an absolute whole number counts as a *fixed name*. (Pasch 1914, 33–34)

On the relevance of the decimal place-value system for the development of arithmetic and for everyday life, Pasch approvingly quotes at length a passage from Kronecker's "Über den Zahlbegriff" (Kronecker 1887, 355).[17] In 1921 Pasch reiterates the importance of numerals and the difference between numbers as things and their names, and here a more structuralist perspective emerges. He writes:

[16] Frege famously considered this to be a problem for a definition of numbers (Frege 1884, §55).

[17] Pasch spent two semesters in 1865–866 in Berlin, attending lectures by Kronecker and Weierstrass. He later mentions these as having exerted a great influence on his thinking about the foundations of mathematics (see Pasch 1930b, 7 and Schlimm 2013b, 189). As far I know, Pasch never expressed any explicit criticism of Kronecker's views of the natural numbers (but, see note 12).

> As we moved along, our starting point, the line $\mathfrak{Z}$, receded entirely into the background. We were no longer concerned with our original choice of *things* to serve as members of the line and, so, as numbers—nor did we care what things were added to the line to accommodate larger and larger numbers. We focused entirely on our need for names and signs for numbers of every size.
>
> Indeed, once the nomenclature for the natural numbers is secured, we can quite disregard whatever things might have gotten us to this point. We need only retain the *names* of these things to perform the task for which the natural numbers were intended: determining whether a collection is equal to another or is greater than it or less. (Pasch 1921, 153)

Although for Pasch the natural numbers continue to be a system of things, this system is not a specific, fixed one, nor does it matter which things we choose. It is tempting to speak in this context of an arbitrary choice of representatives, but that would be misleading: the chosen things do not represent numbers for Pasch, they *are* numbers. Nevertheless, we can see here a form of abstraction from the individual nature of the elements, which is characteristic of a structuralist approach. What matters is only the sequential arrangement, or the structure, of these things, their relations among each other. In addition, it is clear that multiple systems of things can instantiate the natural number structure, which is characterized by the line $\mathfrak{Z}$.

We have seen above that, in his more mature writings, Pasch clearly separates the numbers (which he conceives of as things) from their names, e.g., the decimal place-value numerals. While acknowledging that we can get by with a system of numerals, he does not go so far as identifying the numbers with the numerals themselves, in contrast to some of his contemporaries (e.g., Heine and Thomae, who advocated "formal" theories of arithmetic and were severely criticized by Frege).[18] In order to understand Pasch's account better, it will be useful to compare it to those of two contemporaries that he comments on, namely Alfred Pringsheim and David Hilbert.

In his lectures on number theory (1916), Pringsheim introduces numbers as an infinite "ordered system of signs [*Zeichen*] that satisfies certain rules for their combination" (Pringsheim 1916, vii), mentioning Heine and Helmholtz as other proponents of this view.[19] The simplest such system would be a tally system based on a single primitive sign, "|", but for reasons of practicality Pringsheim decides to use the decimal place-value system as the canonical system of natural numbers (Pringsheim 1916, 7).[20] Thus, Pringsheim does not consider the

[18] For a discussion of criticisms (including those by Frege) of this view, which Detlefsen calls "empiricist formalism," see Detlefsen (2005).

[19] See Heine (1872, 173) and von Helmholtz (1887, 21).

[20] For a critical review, see Hahn (1919).

system of natural numbers to be unique (because different systems of numerals would do), but determined only insofar as it obeys certain rules. Pasch mentions Pringsheim's use of decimal numerals approvingly, but he maintains that his own development of them is "completely different in its nature" (Pasch 1921, 153). How so? For Pasch numbers are not signs (numerals), but those things that the numerals refer to. He also does not want to put the numerals at the beginning of arithmetic, but presents the combinatorial concepts and propositions that underlie the use of numerals. Ultimately, Pasch's interests lie deeper, at the level of the combinatorial origins of numbers.

A few years later, Hilbert also put forward an account of arithmetic based on sequences of signs in his *Neubegründung der Mathematik. Erste Mitteilung* (1922). Soon afterward, Pasch gave a reconstruction of Hilbert's approach to arithmetic in light of his own (Pasch 1924). While he argues that formulas that look like Hilbert's axioms could be derived from his core propositions, Pasch objects to Hilbert's conception of the nature of mathematical objects. Hilbert proclaimed his philosophical standpoint on the foundation of pure mathematics as "at the beginning is the sign [*Zeichen*]", listing as his first definition that "The sign 1 is a number" (Hilbert 1922, 163). First, Pasch disagrees with Hilbert's conception of signs. Hilbert seems to consider signs (and in particular numbers) to be types of inscriptions themselves, whereas for Pasch a sign is an inscription type that denotes a thing. The connection between Hilbert's inscriptions and Pasch's view of numbers is that the former could be considered to be marks, just like the notches on a stick, that could serve as the members of the line $\mathfrak{Z}$ (Pasch 1924, 238). Second, Pasch replaces Hilbert's signs "1" and "+" by the aliases (*Decknamen*) "*e*" and "*u*", such that Hilbert's axioms would correspond to stem propositions, obtained from the core propositions by the process of formalization, i.e., the replacement of meaningful terms by meaningless ones (Pasch 1924, 237, 239–240). In other words, while Hilbert presents a *particular* instance of inscription types as numbers, in Pasch's account it is explicitly recognized that these are just one of many possible instantiations. Thus, by building the possibility of multiple instantiations into his account, Pasch's attitude is clearly more structuralist than Hilbert's, because it also satisfies the second criterion for methodological structuralism, namely envisaging multiple realizations, laid out in section 2.

5. Conclusion

In this chapter two conditions were put forward for a minimal version of methodological structuralism, namely (a) the focus on relational features of systems of objects and (b) envisaging the possibility of having multiple systems that share these relational features. Various factors pushed Pasch toward these two aspects of mathematics. In his work on geometry the quest for rigorous deductions led him to focus on the primitives and relations that are expressed by the axioms and to neglect any other properties that mathematical objects might have. The dualism of projective geometry forced him to accept the possibility that the axioms (stem propositions) can be satisfied by different systems of objects. In Pasch's work on the foundations of arithmetic a structuralist perspective emerged from the fact that the canonical names for numbers, namely the decimal numerals, could refer to any appropriate system of objects. Thus, despite the fact that Pasch maintained an empiricist standpoint, according to which all mathematical knowledge is grounded on experiences of physical objects, he nevertheless came to adopt a methodological structuralism that satisfies both conditions (a) and (b). The further development of structuralism toward a more ontologically oriented position regarding the nature of mathematics went well beyond anything that Pasch would have found acceptable. As Dehn remarks, "The fondness for operating with symbols that have gone far beyond what is intuitable has a mythical-revolutionary character; this was completely foreign to Pasch" (Engel and Dehn 1934, 128). What we see in Pasch's work is that methodological structuralism need not be driven by considerations of abstract structures like those found frequently in modern algebra and that it can be combined successfully with an empiricist philosophy of mathematics.

Acknowledgments

The author would like to thank the participants of the workshops "Origins of Mathematical Structuralism" and "Structuralism in 19th Century Mathematics," held in September 2015 and July 2017 in Vienna for discussions that helped to shape this chapter, and Daniel Lovsted and Georg Schiemer for comments on an earlier version. Work on this chapter was funded by Social Sciences and Humanities Research Council of Canada (SSHRC).

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